

Spiked Models in Wishart Ensemble

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Dong Wang

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Adam Jeffe, Dean of Arts and Sciences

Dissertation Committee:

Mark Adler, Chair

Pierre van Moerbeke

Ira Gessel

Albion Lawrence

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Abstract

Spiked Models in Wishart Ensemble

A dissertation presented to the Faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Dong Wang

The spiked model is an important special case of the Wishart ensemble, and a natural generalization of the white Wishart ensemble. Mathematically, it can be defined on three kinds of variables: the real, the complex and the quaternion. For practical application, we are interested in the limiting distribution of the largest sample eigenvalue.

We first give a new proof of the result of Baik, Ben Arous and P  ch   for the complex spiked model, based on the method of multiple orthogonal polynomials by Bleher and Kuijlaars. Then in the same spirit we present a new result of the rank 1 quaternionic spiked model, proven by combinatorial identities involving quaternionic Zonal polynomials ($\alpha = 1/2$ Jack polynomials) and skew orthogonal polynomials.

We find a phase transition phenomenon for the limiting distribution in the rank 1 quaternionic spiked model as the spiked population eigenvalue increases, and recognize the seemingly new limiting distribution on the critical point as the limiting distribution of the largest sample eigenvalue in the real white Wishart ensemble.

Finally we give conjectures for higher rank quaternionic spiked model and the real spiked model.

Contents

Abstract	v
1 Introduction	1
1.1 Wishart distribution	1
1.2 Spiked models of Wishart ensemble	7
1.3 (Generalized) Zonal polynomials	11
1.4 Statement of Results	17
1.5 Structure of the thesis	23
2 Complex spiked model	25
2.1 Determinantal joint p.d.f. formula	25
2.2 Multiple orthogonal Laguerre polynomials	32
2.3 Proof of theorem 1.1	45
3 Rank 1 quaternionic spiked model	53
3.1 The joint distribution function	53
3.2 The determinantal formula	59
3.3 $S_4(x, y)$ in terms of Laguerre polynomials	63
3.4 Proof of theorem 1.2	70
4 Phase transition phenomenon	82
4.1 Rank 1 complex spiked model	82
4.2 Rank 1 quaternionic spiked model	83
4.3 Conjectures of phase transition in quaternionic and real spiked models	91
5 Asymptotic analysis	94
5.1 Asymptotics of $\psi(p + q\xi)$, $\psi_{r'}(p + q\xi)$, $\tilde{\psi}(p + q\eta)$ and $\tilde{\psi}_{r'}(p + q\eta)$ when $a_{s'} < \gamma^{-1}$	94
5.2 Asymptotics of $\psi_{r'}(p + q\xi)$ and $\tilde{\psi}_{r'}(p + q\eta)$ when $a_{s'} = \gamma^{-1}$	110
5.3 Asymptotics of $\psi(p + q\xi)$, $\psi_{r'}(p + q\xi)$, $\tilde{\psi}(p + q\eta)$ and $\tilde{\psi}_{r'}(p + q\eta)$ when $a_{s'} \leq a$	115
5.4 Asymptotics of Laguerre polynomials and related functions	128

Chapter 1

Introduction

1.1 Wishart distribution

The Wishart distribution is a multivariate generalization of the χ^2 distribution. The χ^2 distribution is defined by the normal distribution. The normal distribution is well known as a distribution of one real variable. However, in this thesis we study statistics of three kinds of variables: real, complex and quaternion. Thus we begin the thesis with a review of the normal distribution of all the three kinds of variables. All results for real variables are standard, see e.g. [22]; for complex variables, see [13]; for quaternion variables, see [4].

1.1.1 Normal distribution

The normal distribution of a real variable with mean μ and variance σ^2 is defined by the probability density function (p.d.f.)

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \quad (1.1)$$

When we study the distribution of a complex variable z , we can view it as two possibly dependent real variables x and y , which are its real and imaginary parts: $z = x + iy$. The definitions of mean and variance is similar to those of real variables:

$$E(z) = E(x) + E(y), \quad (1.2)$$

$$\begin{aligned} \text{var}(z) &= E((z - E(z))\overline{(z - E(z))}) = (E(x^2) - E^2(x)) + (E(y^2) - E^2(y)) \\ &= \text{var}(x) + \text{var}(y). \end{aligned} \quad (1.3)$$

The normal distribution of a complex variable z with mean μ and variance σ^2 is defined by the p.d.f.

$$P(z) = \frac{1}{\pi\sigma^2} e^{-\frac{(x-\Re(\mu))^2 + (y-\Im(\mu))^2}{\sigma^2}} = \frac{1}{\pi\sigma^2} e^{-\frac{|z-\mu|^2}{\sigma^2}}, \quad (1.4)$$

so that x and y are independent real variables in normal distribution with means $\Re(\mu)$ and $\Im(\mu)$ respectively, and identical variance $\sigma^2/2$.

A quaternion variable u has 4 real parts: $u = x + iy + jz + kw$. The definitions of mean and variance is similar to (1.2) and (1.3):¹

$$E(u) = E(x) + iE(y) + jE(z) + kE(w), \quad (1.5)$$

$$\text{var}(u) = E((u - E(u))\overline{(u - E(u))}) = \text{var}(x) + \text{var}(y) + \text{var}(z) + \text{var}(w). \quad (1.6)$$

The normal distribution of a quaternionic variable u with mean μ and variance σ^2 is defined by the p.d.f.

$$P(u) = \frac{1}{\pi^2\sigma^4/4} e^{-\frac{|u-\mu|^2}{\sigma^2/2}}, \quad (1.7)$$

so that the 4 parts, x , y , z and w are independent real variables in normal distribution with means the corresponding parts of μ and identical variance $\sigma^2/4$.

¹In this thesis we use the same notations of conjugation and norm for both complex and quaternion variables. For quaternions, $x + iy + jz + kw = x - iy - jz - kw$ and $|x + iy + jz + kw| = \sqrt{x^2 + y^2 + z^2 + w^2}$.

1.1.2 χ^2 distribution

The χ^2 distribution, like the normal distribution, can be defined for all the three kinds of variables. Let \mathbf{x} be a random variable in normal distribution with mean 0 and variance σ^2 , and we take k independent measurements of \mathbf{x} , with results x_1, \dots, x_k , which are accordingly random variables with identical independent (i.i.d.) $(0, \sigma^2)$ normal distribution. If we define the random variable

$$s = \sum_{j=1}^k |x_j|^2, \quad (1.8)$$

then s/σ^2 is in the χ_k^2 distribution, i.e., chi-square distribution with parameter k .

No matter for what kind of variables χ_k^2 distribution is defined, it is the distribution of a real random variable with support $[0, \infty)$. However, the p.d.f.s of χ_k^2 are not identical for the three kinds of variables:

- The p.d.f. of χ_k^2 for a real variable:

$$P(x) = \frac{(1/2)^{k/2}}{\Gamma(k/2)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}. \quad (1.9)$$

- The p.d.f. of χ_k^2 for a complex variable:

$$P(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}. \quad (1.10)$$

- The p.d.f. of χ_k^2 for a quaternion variable:

$$P(x) = \frac{4^k}{\Gamma(2k)} x^{2k-1} e^{-2x}. \quad (1.11)$$

1.1.3 Multivariate normal distribution

The normal distribution has multivariate versions for all the three kinds of variables. The n -variate random variable is represented by an n -dimensional column vector $X = (x_1, \dots, x_n)^T$, where x_j 's are possibly correlated random variables. The variable in the multivariate normal distribution has a mean μ , which is an n -dimensional column vector, and a covariance matrix Σ , which is a positive defined symmetric, Hermitian or quaternionic Hermitian matrix², depending on which kind of variables we consider. They are given by

$$\mu_j = E(x_j), \quad (1.12)$$

$$\Sigma_{ij} = \text{cov}(x_i, x_j), \quad (1.13)$$

where for any kind of variables,

$$\text{cov}(x, y) = E((x - E(x))\overline{(y - E(y))}), \quad (1.14)$$

with the overline meaning conjugation for complex and quaternion variables and the identity for real variables.

Conversely, μ and Σ determines the p.d.f. of X , by slightly different formulas for the three kinds of variables:

- The p.d.f. of an n -variate real normal variable:

$$P(X) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(X-\mu)^T \Sigma^{-1} (X-\mu)}. \quad (1.15)$$

²The definition of a quaternionic Hermitian matrix $(a_{ij})_{1 \leq i, j \leq N}$ is similar to that of a Hermitian matrix: (1) Diagonal a_{ii} 's are real numbers. (2) Strictly upper-triangular entries a_{ij} 's with $i < j$ are arbitrary quaternions. (3) $a_{ij} = \overline{a_{ji}}$ for strictly lower triangular entries a_{ij} with $i > j$.

- The p.d.f. of an n -variate complex normal variable:

$$P(X) = \frac{1}{\pi^n \det(\Sigma)} e^{-\overline{(X-\mu)}^T \Sigma^{-1} (X-\mu)}. \quad (1.16)$$

- The p.d.f. of an n -variate quaternion normal variable:

$$P(X) = \frac{1}{(\pi/2)^{2n} \det(\Sigma)^2} e^{-2\overline{(X-\mu)}^T \Sigma^{-1} (X-\mu)}. \quad (1.17)$$

We need to take notice that the determinant is not well defined for quaternion matrices due to the noncommutativity of quaternions. Here since Σ is a quaternionic Hermitian matrix with positive real eigenvalues $\sigma_1, \dots, \sigma_n$, we define

$$\det(\Sigma) = \prod_{j=1}^n \sigma_j. \quad (1.18)$$

1.1.4 Wishart distribution

Now we can define the Wishart distribution, which is similar to the χ^2 distribution. Let \mathbf{X} be an N -variate random variable in the normal distribution with mean 0 and covariance matrix Σ , and we take M independent measurements of \mathbf{X} with result X_1, \dots, X_M , which are accordingly random variables with i.i.d. $(0, \Sigma)$ normal distribution. If we let the $N \times M$ matrix X be the juxtaposition of X_j 's: $X = (X_1 :, \dots, : X_M)$, then we say that the N^2 -variate random variable

$$S = \frac{1}{M} X \bar{X}^T \quad (1.19)$$

has the Wishart distribution $W_N(M, \Sigma)$ with M degrees of freedom and covariance matrix Σ .

Given M and Σ , we have explicit formulas of p.d.f.s of S for all the three kinds of

variables. Since the spaces of $N \times N$ real symmetric, Hermitian and quaternionic Hermitian matrices are Euclidean spaces with dimensions $N(N-1)/2$, N^2 and $N(2N-1)$ respectively, we take the usual definition of the measure: For $S = (s_{pq})$ real symmetric, $dS = \prod_{1 \leq q \leq p \leq N} ds_{pq}$. For $S = (s_{pq})$ Hermitian, $dS = \prod_{r=1}^N ds_{rr} \prod_{1 \leq q < p \leq N} d\Re s_{pq} d\Im s_{pq}$. For $S = (s_{pq})$ quaternionic Hermitian and $s_{pq} = x_{pq} + iy_{pq} + jz_{pq} + kw_{pq}$ for off-diagonal entries, $dS = \prod_{r=1}^N s_{rr} \prod_{1 \leq q < p \leq N} dx_{pq} dy_{pq} dz_{pq} dw_{pq}$. To make the support of S be the full positive definite cone of the space of real symmetric matrices, Hermitian matrices or quaternionic Hermitian matrices, we require that $M \geq N$.

- The p.d.f. of the real Wishart distribution $W_N(M, \Sigma)$:

$$P(S)dS = \frac{1}{2^{MN/2} \pi^{N(N-1)/4} (\det \Sigma)^{M/2} \prod_{j=1}^N \Gamma((M-j+1)/2)} e^{-\frac{1}{2} \text{Tr}(\Sigma^{-1}S)} (\det S)^{(M-N-1)/2} dS. \quad (1.20)$$

- The p.d.f. of the complex Wishart distribution $W_N(M, \Sigma)$:

$$P(S)dS = \frac{1}{\pi^{N(N-1)/2} (\det \Sigma)^M \prod_{j=1}^N \Gamma(M-j+1)} e^{-\text{Tr}(\Sigma^{-1}S)} (\det S)^{M-N} dS. \quad (1.21)$$

- The p.d.f. of the quaternion Wishart distribution $W_N(M, \Sigma)$:

$$P(S)dS = \frac{2^{2MN}}{\pi^{N(N-1)} (\det \Sigma)^{2M} \prod_{j=1}^N \Gamma(2(M-j+1))} e^{-2\Re \text{Tr}(\Sigma^{-1}S)} (\det S)^{2(M-N)+1} dS. \quad (1.22)$$

Here we should take note that although $\text{Tr}(\Sigma^{-1}S)$ is automatically real for Σ and S to be real symmetric or complex Hermitian matrices, we need to take the real part explicitly in the quaternionic case due to the noncommutativity.

1.2 Spiked models of Wishart ensemble

In statistics, the eigenvalues $\sigma_1, \dots, \sigma_N$ of the covariance matrix Σ in the multivariate normal distribution are called population eigenvalues. They are of importance in principal component analysis.

Let \mathbf{X} be a centralized N -variate random variable, which means its mean is 0. Under the assumption that \mathbf{X} has the (multivariate) normal distribution, how can we determine its population eigenvalues by results of measurements?

In the $N = 1$ case, it is equivalent to find the variance σ^2 . If we make k independent measurements and get results x_1, \dots, x_k , we have the random variable s defined in (1.8), and it is easy to find that s/k approaches σ^2 almost surely as $k \rightarrow \infty$.

For general N , if we make M measurements and get results X_1, \dots, X_N , we have the $N \times N$ random matrix S defined in (1.19), which is called the sample covariance matrix in statistics. The multivariate counterparts of s are the eigenvalues $\lambda_1, \dots, \lambda_N$ of S , which are called sample eigenvalues in statistics. A celebrated result of Anderson [3] states that if $M \gg N$, the sample eigenvalues are good approximations of the population eigenvalues.

However, if M is not much greater than N , say, both N and M are large, and $M/N = \gamma^2 \geq 1$, then the sample eigenvalues fail to approximate the population eigenvalues. For example, if σ_j 's are identically 1, Marčenko and Pastur proved [20]

Proposition 1.1 (Marčenko-Pastur law). *When $\Sigma = I$, as $M, N \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$, the limiting density of the sample eigenvalues λ_i in the complex Wishart ensemble is given by*

$$\frac{1}{N} \#\{\lambda_j | \lambda_j \leq x\} \rightarrow H(x), \quad (1.23)$$

where $(b_1 = (1 - \gamma^{-1})^2$ and $b_2 = (1 + \gamma^{-1})^2)$

$$H(x) = \begin{cases} 0 & x < b_1 \\ \int_{b_1}^x \frac{\gamma^2}{2\pi t} \sqrt{(t - b_1)(b_2 - t)} & b_1 \leq x \leq b_2 \\ 1 & x > b_2 \end{cases} \quad (1.24)$$

Actually the proof of Marčenko and Pastur is only for real and complex variables. However, their proof can be transplanted to the quaternion case without much difficulty.

In this thesis, if the covariance matrix Σ is given, we call the distribution of the sample eigenvalues $\lambda_1, \dots, \lambda_N$ the Wishart ensemble. It is easy to see that the Wishart ensemble is completely determined by the population eigenvalues $\sigma_1, \dots, \sigma_N$ and the number of measurements M . The Marčenko-Pastur law gives the density of the sample eigenvalues in the $\Sigma = I$ Wishart ensemble, which is commonly called the white Wishart ensemble ³.

The general problem of getting information about σ_j 's from properties of the Wishart ensemble is far too difficult. We begin with the hypothesis testing problem: For Σ and Σ' with different population eigenvalues, can we tell the difference between the corresponding Wishart ensembles? In the simplest case, the eigenvalues of Σ are identically 1, and we get the white Wishart ensemble; most eigenvalues of Σ' are 1 while the other r sample eigenvalues are $1 + \alpha_1, \dots, 1 + \alpha_r$, with α_j 's real numbers greater than -1 , and we call the corresponding Wishart ensemble the spiked model of rank r . Now the question is: Can we tell the spiked model from the white Wishart ensemble?

The density of sample eigenvalues fails to detect the difference, since the proof of

³In the random matrix theory literature, the real white Wishart ensemble is called the Laguerre orthogonal ensemble (LOE), and the complex and quaternionic white Wishart ensembles are called the Laguerre unitary ensemble (LUE) and the Laguerre symplectic ensemble (LSE).

Marčenko and Pastur implies the stronger result [20]:

Proposition 1.2. *Let r be a fixed positive integer and $\alpha_1, \dots, \alpha_r$ fixed real numbers greater than -1 . When $M, N \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$, and the population eigenvalues are $1 + \alpha_1, \dots, 1 + \alpha_r$ and all others identically 1, for all the three kinds of variables, the limiting density of the sample eigenvalues λ_j 's in the rank k spiked model is given by*

$$\frac{1}{N} \#\{\lambda_j | \lambda_j \leq x\} \rightarrow H(x), \quad (1.25)$$

where $H(x)$ is defined in (1.24), the same as that in the white Wishart ensemble.

However, if some population eigenvalues are large, the limiting distribution of the largest sample eigenvalue may change. First, we have a complete result for the limiting distribution of the largest eigenvalue in the white Wishart ensemble. Unlike the limiting density, these limiting distributions for the three kinds of variables are different [9], [15], [16].

Proposition 1.3 (GOE, GUE and GSE Tracy-Widom distributions). *When $\Sigma = I$, as $M, N \rightarrow \infty$ such that $M/N \rightarrow \gamma^2 \geq 1$, the largest sample eigenvalue $\max(\lambda)$ approaches $(1 + \gamma^{-1})^2$ almost surely for all the three kinds of variables. However, the limiting distributions for these three kinds of variables are different.*

- *For real variables, the limiting distribution of $\max(\lambda)$ is the GOE Tracy-Widom distribution, after proper rescaling:*⁴

$$\lim_{M \rightarrow \infty} \mathbb{P} \left((\max(\lambda) - (1 + \gamma^{-1})^2) \cdot \frac{\gamma M^{2/3}}{(1 + \gamma)^{4/3}} < T \right) = F_{\text{GOE}}(T). \quad (1.26)$$

- *For complex variables, the limiting distribution of $\max(\lambda)$ is the GUE Tracy-*

⁴By $M \rightarrow \infty$, we mean “ $M, N \rightarrow \infty$ and $M/N = \gamma^2$ ”.

Widom distribution, after proper rescaling:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left((\max(\lambda) - (1 + \gamma^{-1})^2) \cdot \frac{\gamma M^{2/3}}{(1 + \gamma)^{4/3}} < T \right) = F_{\text{GUE}}(T). \quad (1.27)$$

- *For quaternion variables, the limiting distribution of $\max(\lambda)$ is the GSE Tracy-*

Widom distribution, after proper rescaling:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left((\max(\lambda) - (1 + \gamma^{-1})^2) \cdot \frac{\gamma(2M)^{2/3}}{(1 + \gamma)^{4/3}} < T \right) = F_{\text{GSE}}(T). \quad (1.28)$$

We have explicit formulas for these probability functions [28], [29]:

$$F_{\text{GOE}}(\xi) = e^{-\frac{1}{2} \int_{\xi}^{\infty} (x-\xi) q^2(x) dx} e^{-\frac{1}{2} \int_{\xi}^{\infty} q(x) dx}, \quad (1.29)$$

$$F_{\text{GUE}}(\xi) = e^{-\int_{\xi}^{\infty} (x-\xi) q^2(x) dx}, \quad (1.30)$$

$$F_{\text{GSE}}(\xi) = \frac{1}{2} e^{-\frac{1}{2} \int_{\xi}^{\infty} (x-\xi) q^2(x) dx} \left(e^{-\frac{1}{2} \int_{\xi}^{\infty} q(x) dx} + e^{\frac{1}{2} \int_{\xi}^{\infty} q(x) dx} \right), \quad (1.31)$$

where $q(x)$ is a solution to the Painlevé equation

$$q''(x) = xq(x) + 2q^3(x), \quad (1.32)$$

with

$$q(x) \sim \text{Ai}(x) \quad \text{as } x \rightarrow +\infty. \quad (1.33)$$

Here $\text{Ai}(x)$ is the Airy function, whose definition will be given in (1.62).

It is worth noticing that although all previous statements for the three kinds of variables are parallel, the three-fold symmetry breaks down and our statements for the three kinds of variables are going to bifurcate. Despite their similarity in analytic form, F_{GUE} is most naturally defined by a Fredholm determinant while F_{GOE} and F_{GSE} are most naturally defined by Fredholm Pfaffians, and the derivations of F_{GOE} and

F_{GSE} are somehow similar to each other, and requires more work than the derivation of F_{GUE} [21].

Now the goal is to get the limiting distribution of the largest eigenvalue in spiked models of the three kinds of variables. For the complex variables, Baik, Ben Arous and P     got the complete result for any finite rank k [6]. In this thesis we derive their result by a different approach and are going to get that limiting distribution for the rank 1 quaternionic spiked model. It is desirable to find the counterpart limiting distribution for the rank 1 real spiked model. However, it seems that the symmetry between real and quaternion variables breaks again, and such a result is unattainable by the method in this thesis.

1.3 (Generalized) Zonal polynomials

From the p.d.f.s of the Wishart distributions (1.20)–(1.22) for the three kinds of variables, we can get the p.d.f.s of sample eigenvalues $\lambda = (\lambda_1, \dots, \lambda_N)$ by the Weyl integration formula, or more directly, by calculating Jacobians. Since we do not need explicit formulas of the normalization constants, we simply write “ C ” from now on. The derivation for real variables is in [22]

- The p.d.f. of the sample eigenvalues in the real Wishart ensemble:

$$P(\lambda) = \frac{1}{C} |V(\lambda)| \prod_{j=1}^N \lambda_j^{(M-N-1)/2} \int_{O(N)} e^{-\frac{M}{2} \text{Tr}(\Sigma^{-1} O S O^{-1})} dO. \quad (1.34)$$

- The p.d.f. of the sample eigenvalues in the complex Wishart ensemble:

$$P(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} \int_{U(N)} e^{-M \text{Tr}(\Sigma^{-1} U S U^{-1})} dU. \quad (1.35)$$

- The p.d.f. of the sample eigenvalues in the quaternion Wishart ensemble:

$$P(\lambda) = \frac{1}{C} V(\lambda)^4 \prod_{j=1}^N \lambda_j^{2(M-N)+1} \int_{Sp(N)} e^{-2M\Re \text{Tr}(\Sigma^{-1} Q S Q^{-1})} dQ. \quad (1.36)$$

Here $V(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)$ is the Vandermonde, and the integrals in (1.34)–(1.36) are over orthogonal, unitary and compact symplectic groups with Haar measures respectively. To go further in our analysis, we need to evaluate these integrals.

It is well known among statisticians that we can expand the integral in (1.34) by Zonal polynomials [22]. To define Zonal polynomials, and their counterparts for complex and quaternion variables, we need some preliminary definitions.

Definition 1.1. [19] A partition κ of k is a sequence $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_l)$ where $\kappa_j \geq 0$ are weakly decreasing and $\sum_{j=1}^l \kappa_j = k$. We denote this by $\kappa \vdash k$.

For example, $\kappa = (2, 2, 1)$ is a partition of 5. The number of nonzero parts of κ is called the length of κ , denoted as $l(\kappa)$. If we drop the weakly decreasing condition, we call the sequence a general partition.

If κ and κ' are two general partitions of k , we say $\kappa < \kappa'$ if for some index j , $\kappa_i = \kappa'_i$ for $i < j$ and $\kappa_j < \kappa'_j$. For example,

$$(2, 1, 1, 1) < (2, 2, 1) < (3, 2). \quad (1.37)$$

If $\kappa \vdash k$ is a general partition with $l(\kappa) = l$, we define the monomial of degree k

$$x^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \dots x_l^{\kappa_l} \quad (1.38)$$

and say $x^{\kappa'}$ is of higher weight than x^κ if $\kappa' > \kappa$.

We need another definition of Laplacians $\Delta_x^{\mathbb{R}}$, $\Delta_x^{\mathbb{C}}$ and $\Delta_x^{\mathbb{H}}$ [25]:

Definition 1.2. For N variables $x = (x_1, \dots, x_N)$, we define

$$\Delta_x^{\mathbb{R}} = \sum_{j=1}^N x_j^2 \frac{\partial^2}{\partial^2 x_j^2} + \sum_{\substack{i=1 \\ i \neq j}}^N \frac{x_j^2}{x_j - x_i} \frac{\partial}{\partial x_j}, \quad (1.39)$$

$$\Delta_x^{\mathbb{C}} = \sum_{j=1}^N x_j^2 \frac{\partial^2}{\partial^2 x_j^2} + 2 \sum_{\substack{i=1 \\ i \neq j}}^N \frac{x_j^2}{x_j - x_i} \frac{\partial}{\partial x_j}, \quad (1.40)$$

$$\Delta_x^{\mathbb{H}} = \sum_{j=1}^N x_j^2 \frac{\partial^2}{\partial^2 x_j^2} + 4 \sum_{\substack{i=1 \\ i \neq j}}^N \frac{x_j^2}{x_j - x_i} \frac{\partial}{\partial x_j}. \quad (1.41)$$

Now we can give a definition of Zonal polynomials and their counterparts, complex Zonal polynomials and quaternionic Zonal polynomials [18].

Definition 1.3. For N variables $x = (x_1, \dots, x_N)$, a nonnegative integer k and a partition $\kappa \vdash k$, we have the unique Zonal polynomial $Z_\kappa(x)$, complex Zonal polynomial $C_\kappa(x)$ and quaternionic Zonal polynomial $Q_\kappa(x)$, which are all symmetric, homogeneous polynomials of degree k in x_j 's such that

- The highest weight term in $Z_\kappa(x)$ ($C_\kappa(x)$, $Q_\kappa(x)$) is x^κ .
- $Z_\kappa(x)$ ($C_\kappa(x)$, $Q_\kappa(x)$) is an eigenfunction of the Laplacian $\Delta_x^{\mathbb{R}}$ ($\Delta_x^{\mathbb{C}}$, $\Delta_x^{\mathbb{H}}$).
-

$$\sum_{\kappa \vdash k} Z_\kappa(x) = \sum_{\kappa \vdash k} C_\kappa(x) = \sum_{\kappa \vdash k} Q_\kappa(x) = (x_1 + \dots + x_N)^k. \quad (1.42)$$

Form the highest weight property of $Z_\kappa(x)$ ($C_\kappa(x)$, $Q_\kappa(x)$), we have

Fact 1.1. For any N and $\kappa \vdash k$,

$$Z_\kappa(x) = C_\kappa(x) = Q_\kappa(x) = 0 \quad \text{if} \quad l(\kappa) > N. \quad (1.43)$$

Latter we apply the notation $Z_\kappa(X)$ to mean $Z_\kappa(x_1, \dots, x_N)$ if X is an $N \times N$ matrix with eigenvalues x_1, \dots, x_N , and similarly for $C_\kappa(X)$ and $Q_\kappa(X)$.

It can be derived from their eigenfunction property that [22], [18], [19], [14]

Proposition 1.4. *Given X and Y to be $N \times N$ symmetric matrices, Hermitian matrices or quaternionic Hermitian matrices, and I_N to be the $N \times N$ identity matrix, we have*

$$\int_{O(N)} Z_\kappa(XOYO^{-1})dO = \frac{Z_\kappa(X)Z_\kappa(Y)}{Z_\kappa(I_N)}, \quad (1.44)$$

$$\int_{U(N)} C_\kappa(XUYU^{-1})dU = \frac{C_\kappa(X)C_\kappa(Y)}{C_\kappa(I_N)}, \quad (1.45)$$

$$\int_{Sp(N)} \Re Q_\kappa(XQYQ^{-1})dQ = \frac{Q_\kappa(X)Q_\kappa(Y)}{Q_\kappa(I_N)}. \quad (1.46)$$

Then by (1.42) we have

$$\begin{aligned} \int_{O(N)} e^{\rho \text{Tr}(XOYO^{-1})} dO &= \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \int_{O(N)} \text{Tr}(XOYO^{-1})^k dO \\ &= \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \int_{O(N)} Z_\kappa(XOYO^{-1}) dO \\ &= \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{Z_\kappa(X)Z_\kappa(Y)}{Z_\kappa(I_N)}, \end{aligned} \quad (1.47)$$

and similarly

$$\int_{U(N)} e^{\rho \text{Tr}(XUYU^{-1})} dU = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{C_\kappa(X)C_\kappa(Y)}{C_\kappa(I_N)}, \quad (1.48)$$

$$\int_{Sp(N)} e^{\rho \Re \text{Tr}(XQYQ^{-1})} dQ = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{Q_\kappa(X)Q_\kappa(Y)}{Q_\kappa(I_N)}. \quad (1.49)$$

Now we can state the series formulas for joint p.d.f.s for the three kinds of variables of the Wishart ensemble:

Proposition 1.5. *Let the centralized N -variate normal random variable have the*

covariance matrix Σ with population eigenvalues $\sigma_1, \dots, \sigma_N$, and the number of measurements be $M \geq N$. The p.d.f. of the sample eigenvalues $\lambda = (\lambda_1, \dots, \lambda_N)$ in the Wishart ensemble is

- For the real variable case

$$P(\lambda) = \frac{1}{C} |V(\lambda)| \prod_{j=1}^N \lambda_j^{(M-N-1)/2} \sum_{k=0}^{\infty} \frac{(-M/2)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{Z_{\kappa}(\sigma_1^{-1}, \dots, \sigma_N^{-1}) Z_{\kappa}(\lambda_1, \dots, \lambda_N)}{Z_{\kappa}(I_N)}. \quad (1.50)$$

- For the complex variable case

$$P(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} \sum_{k=0}^{\infty} \frac{(-M)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{C_{\kappa}(\sigma_1^{-1}, \dots, \sigma_N^{-1}) C_{\kappa}(\lambda_1, \dots, \lambda_N)}{C_{\kappa}(I_N)}. \quad (1.51)$$

- For the quaternion variable case

$$P(\lambda) = \frac{1}{C} V(\lambda)^4 \prod_{j=1}^N \lambda_j^{2(M-N)+1} \sum_{k=0}^{\infty} \frac{(-2M)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{Q_{\kappa}(\sigma_1^{-1}, \dots, \sigma_N^{-1}) Q_{\kappa}(\lambda_1, \dots, \lambda_N)}{Q_{\kappa}(I_N)}. \quad (1.52)$$

Remark 1.1. The Zonal, complex Zonal and quaternionic Zonal polynomials are Jack polynomials with the parameter $\alpha = 2, 1$ and $1/2$ [19]. In particular, complex Zonal polynomials ($\alpha = 1$ Jack polynomials) are essentially Schur polynomials, see (2.1).

Although in formulas (1.50)–(1.52) we get rid of integrals over Lie groups, the number of degree k terms grows very fast as k increases, since the only restriction $l(k) \leq N$ is rather weak when we consider the large N . For the general Wishart ensemble these formulas are still impractical. However, they can be much more powerful in spiked models.

Since in the spiked model, lots of σ_j 's are identically 1, we can shift coordinates to make them identically 0, and then by fact 1.1, for rank r spiked model we need only consider (complex, quaternionic) Zonal polynomials with index $l(\kappa) \leq r$, since after the coordinate shift, $N - r$ variables are 0 and the N -variable polynomials is equivalent to an r -variable one. For real variables the procedure is (eigenvalues of S are $\lambda_1, \dots, \lambda_N$, and eigenvalues of Σ are $\alpha_1, \dots, \alpha_r, 1, \dots, 1$)

$$\begin{aligned}
& \int_{O(N)} e^{-\frac{M}{2} \text{Tr}(\Sigma^{-1}OSO^{-1})} dO \\
&= \int_{O(N)} e^{-\frac{M}{2} \text{Tr}(IOSO^{-1})} e^{\frac{M}{2} \text{Tr}((I-\Sigma^{-1})OSO^{-1})} dO \\
&= \prod_{j=1}^N e^{-\frac{M}{2} \lambda_j} \int_{O(N)} e^{\frac{M}{2} \text{Tr}((I-\Sigma^{-1})OSO^{-1})} dO \\
&= \prod_{j=1}^N e^{-\frac{M}{2} \lambda_j} \sum_{k=0}^{\infty} \frac{(M/2)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq r}} \frac{Z_{\kappa}(\frac{\alpha_1}{1+\alpha_1}, \dots, \frac{\alpha_r}{1+\alpha_r}) Z_{\kappa}(\lambda_1, \dots, \lambda_N)}{Z_{\kappa}(I_N)},
\end{aligned} \tag{1.53}$$

since r eigenvalues of $I - \Sigma^{-1}$ are $\frac{\alpha_1}{1+\alpha_1}, \dots, \frac{\alpha_r}{1+\alpha_r}$ and the other $N - r$ eigenvalues are 0. In this way we can simplify formulas for joint p.d.f.s of sample eigenvalues for the three kinds of variables in the spiked model:

Proposition 1.6. *Let the centralized N -variate normal random variable have the covariance matrix Σ with r population eigenvalues $1 + \alpha_1, \dots, 1 + \alpha_r$ and the other $N - r$ population eigenvalues identically 1, and the number of measurements be $M \geq N$. The p.d.f. of the sample eigenvalues $\lambda = (\lambda_1, \dots, \lambda_N)$ in the rank r spiked model*

is

- For the real variable case

$$P(\lambda) = \frac{1}{C} |V(\lambda)| \prod_{j=1}^N \lambda_j^{(M-N-1)/2} e^{-\frac{M}{2} \lambda_j} \sum_{k=0}^{\infty} \frac{(-M/2)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq r}} \frac{Z_{\kappa}(\frac{\alpha_1}{1+\alpha_1}, \dots, \frac{\alpha_r}{1+\alpha_r}) Z_{\kappa}(\lambda_1, \dots, \lambda_N)}{Z_{\kappa}(I_N)}. \quad (1.54)$$

- For the complex variable case

$$P(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} e^{-M \lambda_j} \sum_{k=0}^{\infty} \frac{(-M)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{C_{\kappa}(\frac{\alpha_1}{1+\alpha_1}, \dots, \frac{\alpha_r}{1+\alpha_r}) C_{\kappa}(\lambda_1, \dots, \lambda_N)}{C_{\kappa}(I_N)}. \quad (1.55)$$

- For the quaternion variable case

$$P(\lambda) = \frac{1}{C} V(\lambda)^4 \prod_{j=1}^N \lambda_j^{2(M-N)+1} e^{-2M \lambda_j} \sum_{k=0}^{\infty} \frac{(-2M)^k}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq N}} \frac{Q_{\kappa}(\frac{\alpha_1}{1+\alpha_1}, \dots, \frac{\alpha_r}{1+\alpha_r}) Q_{\kappa}(\lambda_1, \dots, \lambda_N)}{Q_{\kappa}(I_N)}. \quad (1.56)$$

1.4 Statement of Results

1.4.1 Complex spiked model

To demonstrate the result, we need the language of Fredholm determinant. If $K(x, y)$ is the kernel of an integral operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, then the Fredholm determinant of the integral operator, which we represent by the same notation as its kernel,

is [12]

$$\begin{aligned} \det(I - K(x, y)) = & \\ & 1 - \frac{1}{1!} \int_{-\infty}^{\infty} K(x_1, x_1) dx_1 + \frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{vmatrix} K(x_1, x_1) & K(x_1, x_2) \\ K(x_2, x_1) & K(x_2, x_2) \end{vmatrix} dx_1 dx_2 - \dots \\ & + \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \begin{vmatrix} K(x_1, x_1) & \dots & K(x_1, x_n) \\ \vdots & \dots & \vdots \\ K(x_n, x_1) & \dots & K(x_n, x_n) \end{vmatrix} dx_1 \dots dx_n + \dots \end{aligned} \quad (1.57)$$

Then we can state the theorem for the complex spiked model [6]

Theorem 1.1. *In the rank r complex spiked model, let non-trivial population eigenvalues be $1 + a_1 < \dots < 1 + a_s$ with multiplicities respectively r_1, \dots, r_s , so that $\sum_{j=1}^s r_j = r$.*

1. *If $-1 < a_s < \gamma^{-1}$, then the distribution of the largest sample eigenvalue is the same as that of the complex white Wishart ensemble in proposition 1.3.*
2. *If $a_s = \gamma^{-1}$, then the limit and the fluctuation scale are the same as those of the complex white Wishart ensemble, but the distribution function is*

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\left(\max(\lambda) - \left(\frac{\gamma + 1}{\gamma} \right)^2 \right) \cdot \frac{\gamma M^{2/3}}{(\gamma + 1)^{4/3}} < T \right) = F_{\text{GUE}_{r_s}}(T). \quad (1.58)$$

3. *If $a_s = a > \gamma^{-1}$, then the limit and the fluctuation scale are changed as well as the distribution function, which is a finite GUE distribution*

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\left(\max(\lambda) - (1 + a) \left(1 + \frac{1}{\gamma^2 a} \right) \right) \cdot \frac{\sqrt{M}}{(1 + a) \sqrt{1 - \frac{1}{\gamma^2 a^2}}} < T \right) = G_{r_s}(T). \quad (1.59)$$

We need to explain the distribution function $F_{\text{GUE}r_s}$ and G_{r_s} . First, let us revisit the distribution F_{GUE} , which can be defined by (We abbreviate the indicator function $\chi_{[T,\infty)}(x)$ as $\chi(x)$.) [28]

$$F_{\text{GUE}}(T) = \det (1 - \chi(\xi) K_{\text{Airy}}(\xi, \eta) \chi(\eta)), \quad (1.60)$$

where the kernel K_{Airy} is given by the Airy function

$$K_{\text{Airy}}(\xi, \eta) = \int_0^\infty \text{Ai}(x+t) \text{Ai}(y+t) dt, \quad (1.61)$$

with [1]

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} e^{ixz + i\frac{z^3}{3}} dz. \quad (1.62)$$

The integral sign $\int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}}$ means the integral is along an infinite arc from the direction $\infty e^{5\pi i/6}$ to the direction $\infty e^{\pi i/6}$. (We borrow the notation from the real projective geometry on \mathbb{RP}^2 .) It is easy to see that as $x \rightarrow +\infty$, $|\text{Ai}(x)| \rightarrow 0$ faster than any exponential decay. The equivalency of the Fredholm determinant representation and the formula (1.30) is established by Tracy and Widom.

$F_{\text{GUE}t}$ with $t = 1, 2, \dots$ are variations of F_{GUE} , and are defined as [6]

$$F_{\text{GUE}t}(T) = \det \left(1 - \chi(\xi) \left(K_{\text{Airy}}(\xi, \eta) + \sum_{j=1}^t t^{(j)}(\xi) s^{(j)}(\eta) \right) \chi(\eta) \right), \quad (1.63)$$

where $s^{(j)}$ and $t^{(j)}$ are variations of the Airy function, with

$$s^{(j)}(x) = \frac{1}{2\pi} \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} e^{ixz + i\frac{z^3}{3}} \frac{1}{(iz)^j} dz, \quad (1.64)$$

$$t^{(j)}(x) = \frac{1}{2\pi} \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} e^{ixz + i\frac{z^3}{3}} (-iz)^{j-1} dz, \quad (1.65)$$

and we require that the infinite arc for the integral of $s^{(j)}$ is below the pole $z = 0$. We can prove that as $x \rightarrow +\infty$, $|t^{(j)}(x)| \rightarrow 0$ faster than any exponential decay, and $s^{(j)}$ grows slower than any exponential growth. Especially,

$$s^{(1)}(x) = 1 - \int_x^\infty \text{Ai}(t) dt, \quad (1.66)$$

$$t^{(1)}(x) = \text{Ai}(x). \quad (1.67)$$

G_t with $t = 1, 2, \dots$ are defined as

$$G_t(T) = \det \left(1 - \chi(\xi) \left(\sum_{j=0}^{t-1} \frac{1}{j! \sqrt{2\pi}} H_j(\xi) e^{-\frac{\xi^2}{4}} H_j(\eta) e^{-\frac{\eta^2}{4}} \right) \chi(\eta) \right), \quad (1.68)$$

where H_j 's are Hermite polynomials [27], with $\deg(H_j) = j$ and

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) e^{-\frac{x^2}{2}} dx = j! \sqrt{2\pi} \delta_{ij}. \quad (1.69)$$

Remark 1.2. Since $H_0(x) = 1$ is a constant, we can easily see that G_1 is the Gaussian distribution

$$G_1(T) = \int_{-\infty}^T \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt. \quad (1.70)$$

Remark 1.3. We call F_{GUE} , $F_{\text{GUE}t}$ and G_t distribution functions, because they are monotone increasing, as $T \rightarrow +\infty$, the values of these functions approach 1, and as $T \rightarrow -\infty$, the values approach 0. Their monotonicity is ensured by their definitions. In the $T \rightarrow +\infty$ direction, we can easily verify the limit property from the determinantal representation. However, in the $T \rightarrow -\infty$ direction, it is a more serious problem.

Since G_t is defined by a finite rank kernel with relatively simple functions, we can verify $\lim_{T \rightarrow -\infty} G_t = 0$ by direct calculation. For F_{GUE} , the analytic formula (1.30), whose derivation from the determinantal formula (1.60) is highly non-trivial, yields

the limit result as $T \rightarrow -\infty$. For $F_{\text{GUE}t}$, we refer to Baik's result [5].

1.4.2 Rank 1 quaternionic spiked model

To state the result for the rank 1 quaternionic spiked model, we need the definition of Fredholm determinant for a matrix integral operator. If K is a matrix integral operator from $L^2(\mathbb{R})^n$ to $L^2(\mathbb{R})^n$, i.e., $K = (K_{ij}(x, y))_{1 \leq i, j \leq n}$, with $K_{ij}(x, y)$ an integral operator from $L^2(\mathbb{R})^n$ to $L^2(\mathbb{R})^n$, then

$$\det(I - K) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{\substack{0 \leq r_j \leq m \\ \sum_{j=1}^n r_j = m}} \frac{1}{r_1! \cdots r_n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{r_1} dx_j^{(1)} \cdots \prod_{j=1}^{r_n} dx_j^{(n)} \det \left(K_{kl}(x_i^{(k)}, x_j^{(l)})_{\substack{1 \leq i \leq r_k \\ 1 \leq l \leq r_l}} \right)_{1 \leq k, l \leq m}. \quad (1.71)$$

Then we can state the theorem for the rank 1 quaternionic spiked model

Theorem 1.2. *In the rank 1 quaternionic spiked model,*

1. *If $-1 < a < \gamma^{-1}$, then the distribution of the largest sample eigenvalue is the same as that of the quaternionic white Wishart ensemble in proposition 1.3.*
2. *If $a = \gamma^{-1}$, then the limit and the fluctuation scale are the same as those of the quaternionic white Wishart ensemble, but the distribution function is*

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\left(\max(\lambda) - \left(\frac{\gamma + 1}{\gamma} \right)^2 \right) \cdot \frac{\gamma(2M)^{2/3}}{(1 + \gamma)^{4/3}} < T \right) = F_{\text{GSE}1}(T). \quad (1.72)$$

3. *If $a > \gamma^{-1}$, then the limit and the fluctuation scale are changed as well as the*

distribution function, which is a Gaussian:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\left(\max(\lambda) - (a+1) \left(1 + \frac{1}{\gamma^2 a} \right) \right) \cdot \frac{\sqrt{2M}}{(a+1) \sqrt{1 - \frac{1}{\gamma^2 a^2}}} < T \right) = \int_{-\infty}^T \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt. \quad (1.73)$$

Here F_{GSE1} is a variation of F_{GSE} , and we first give a definition of F_{GSE} by Fredholm determinant of a matrix integral operator [11]

$$F_{\text{GSE}}(T) = \sqrt{\det(1 - \widehat{P}(\xi, \eta))}, \quad (1.74)$$

and

$$\widehat{P}(\xi, \eta) = \chi(\xi) \begin{pmatrix} \widehat{S}_4(\xi, \eta) & \widehat{SD}_4(\xi, \eta) \\ \widehat{IS}_4(\xi, \eta) & \widehat{S}_4(\eta, \xi,) \end{pmatrix} \chi(\eta), \quad (1.75)$$

where

$$\widehat{S}_4(\xi, \eta) = \frac{1}{2} K_{\text{Airy}}(\xi, \eta) - \frac{1}{4} \text{Ai}(\xi) \int_{\eta}^{\infty} \text{Ai}(t) dt, \quad (1.76)$$

$$\widehat{SD}_4(\xi, \eta) = -\frac{1}{2} \frac{\partial}{\partial \eta} K_{\text{Airy}}(\xi, \eta) - \frac{1}{4} \text{Ai}(\xi) \text{Ai}(\eta), \quad (1.77)$$

$$\widehat{IS}_4(\xi, \eta) = -\frac{1}{2} \int_{\xi}^{\infty} K_{\text{Airy}}(t, \eta) dt + \frac{1}{4} \int_{\xi}^{\infty} \text{Ai}(t) dt \int_{\eta}^{\infty} \text{Ai}(t) dt. \quad (1.78)$$

The equivalency of (1.74) and the formula (1.31) is established in [29] and [31].

Now we can define F_{GSE1} as

$$F_{\text{GSE1}}(T) = \sqrt{\det(1 - \overline{\overline{P}}(\xi, \eta))}, \quad (1.79)$$

and

$$\overline{\overline{P}}(\xi, \eta) = \chi(\xi) \begin{pmatrix} \overline{\overline{S}}_4(\xi, \eta) & \overline{\overline{SD}}_4(\xi, \eta) \\ \overline{\overline{IS}}_4(\xi, \eta) & \overline{\overline{S}}_4(\eta, \xi,) \end{pmatrix} \chi(\eta), \quad (1.80)$$

spiked model, which is from G_{GSE} to Gaussian via F_{GOE} . We also give conjectures for more general phase transition phenomena.

Chapter 2

Complex spiked model

In this chapter, we consider the complex Wishart ensemble unless otherwise stated.

2.1 Determinantal joint p.d.f. formula

For the complex Wishart ensemble, we have an advantage that the complex Zonal polynomial $C_\kappa(x)$ is the same as the Schur polynomial $s_\kappa(x)$ up to a constant multiple.

To be precise, we have [26]

$$C_\kappa(x) = \frac{k!}{H(\kappa)} s_\kappa(x), \quad (2.1)$$

where $H(\kappa)$ is the product of hook lengths of κ . If $\kappa = (\kappa_1, \kappa_2, \dots)$ and $l(\kappa) = l$, then

$$H(\kappa) = \prod_{i=1}^l \prod_{j=1}^{\kappa_i} (\text{arm}_\kappa(i, j) + \text{leg}_\kappa(i, j) + 1), \quad (2.2)$$

where

$$\text{arm}_\kappa(i, j) = \kappa_i - j, \quad (2.3)$$

$$\text{leg}_\kappa(i, j) = \min\{i' \mid \kappa_{i'} < j\}. \quad (2.4)$$

The nomenclature of arm_κ and leg_κ is most clearly shown by the Young diagram of the partition κ . For example, in the Young diagram of the partition $\kappa = (4, 3, 3, 2, 1)$ which is drawn in figure 2.1, $\text{arm}_\kappa(2, 1)$ is the number of squares to the right of the 1-st square in the 2-nd row, and $\text{leg}_\kappa(2, 1)$ is the number of squares below it. Usually we call $\text{arm}_\kappa(i, j) + \text{leg}_\kappa(i, j) + 1 = h_\kappa(i, j)$ the hook length of the (i, j) square in the Young diagram of the partition κ .

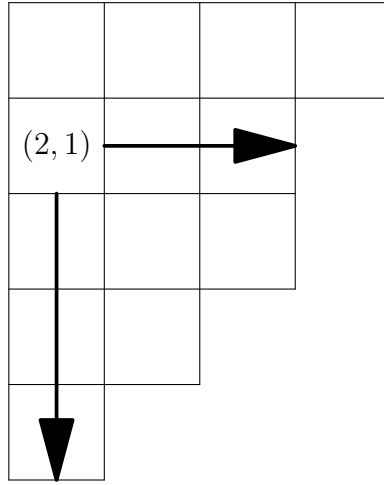


Figure 2.1: Young tableau of partition $(4, 3, 3, 2, 1)$

Let us consider the rank 1 spiked model first. We assume the N population eigenvalues are $(N - 1)$ 1's and the other one $1 + a$. With M measurements, by (1.55), the joint p.d.f. of sample eigenvalues is

$$P(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j} \sum_{k=0}^{\infty} \frac{M^k}{k!} \frac{C_{(k)}(\frac{a}{1+a}) C_{(k)}(\lambda_1, \dots, \lambda_N)}{C_{(k)}(I_N)}. \quad (2.5)$$

Since we require the index $\kappa \vdash k$ of complex Zonal polynomials to satisfy $l(\kappa) \leq 1$, for any k , there is only one qualified partition $\kappa = (k)$. Therefore the joint p.d.f. of sample eigenvalues for the rank 1 spiked model is much simplified.

Here we notice that $H((k)) = k!$, so that $C_{(k)}(x) = s_{(k)}(x)$. We have

$$s_{(k)}\left(\frac{a}{1+a}\right) = \left(\frac{a}{1+a}\right)^k, \quad (2.6)$$

$$s_{(k)}(I_N) = \frac{(N+k-1)!}{(N-1)!k!}. \quad (2.7)$$

Therefore

$$P(\lambda) = \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j} \sum_{k=0}^{\infty} \frac{\left(\frac{a}{1+a}M\right)^k}{(N+k-1)!} s_{(k)}(\lambda_1, \dots, \lambda_N). \quad (2.8)$$

Schur polynomials have the well known determinantal representation [19]: For any partition $\kappa = (\kappa_1, \kappa_2, \dots)$ with $l(\kappa) = l$,

$$s_{\kappa}(x_1, \dots, x_N) = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \dots & \vdots \\ x_1^{N-l-1} & x_2^{N-l-1} & \dots & x_N^{N-l-1} \\ x_1^{N-l+\kappa_l} & x_2^{N-l+\kappa_l} & \dots & x_N^{N-l+\kappa_l} \\ \vdots & \vdots & \dots & \vdots \\ x_1^{N-1+\kappa_1} & x_2^{N-1+\kappa_1} & \dots & x_N^{N-1+\kappa_1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \dots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{vmatrix}}, \quad (2.9)$$

where the denominator is the Vandermonde and the numerator is different from the Vandermonde only at the last l rows, with the power of entries of the j -th last row increased by κ_j .

We have

$$s_{(k)}(\lambda_1, \dots, \lambda_N) = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \lambda_N^{N-2} \\ \lambda_1^{N-1+k} & \lambda_2^{N-1+k} & \dots & \lambda_N^{N-1+k} \end{vmatrix}}{V(\lambda)}, \quad (2.10)$$

and

$$\sum_{k=0}^{\infty} \frac{\left(\frac{a}{1+a}M\right)^k}{(N+k-1)!} s_{(k)}(\lambda_1, \dots, \lambda_N) = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \lambda_N^{N-2} \\ \tilde{e}\left(\frac{a}{1+a}M\lambda_1\right) & \tilde{e}\left(\frac{a}{1+a}M\lambda_2\right) & \dots & \tilde{e}\left(\frac{a}{1+a}M\lambda_N\right) \end{vmatrix}}{V(\lambda)}, \quad (2.11)$$

where

$$\begin{aligned} \tilde{e}\left(\frac{a}{1+a}M\lambda_j\right) &= \sum_{k=0}^{\infty} \frac{1}{(N+k-1)!} \left(\frac{a}{1+a}M\right)^k \lambda_j^{N+k-1} \\ &= \left(\frac{a}{1+a}M\right)^{-(N-1)} \left(e^{\frac{a}{1+a}M\lambda_j} - \sum_{k=0}^{N-2} \frac{1}{k!} \left(\frac{a}{1+a}M\lambda_j\right)^k \right). \end{aligned} \quad (2.12)$$

By row operations, we can change the term $\tilde{e}\left(\frac{a}{1+a}M\lambda_j\right)$ in (2.11) into $\left(\frac{a}{1+a}M\right)^{-(N-1)} e^{\frac{a}{1+a}M\lambda_j}$,

and we have

$$\begin{aligned}
 P(\lambda) &= \frac{1}{C} V(\lambda)^2 \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j} \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \lambda_N^{N-2} \\ e^{\frac{a}{1+a}M\lambda_1} & e^{\frac{a}{1+a}M\lambda_2} & \dots & e^{\frac{a}{1+a}M\lambda_N} \end{vmatrix}}{V(\lambda)} \\
 &= \frac{1}{C} V(\lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \lambda_N^{N-2} \\ e^{\frac{a}{1+a}M\lambda_1} & e^{\frac{a}{1+a}M\lambda_2} & \dots & e^{\frac{a}{1+a}M\lambda_N} \end{vmatrix} \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j}.
 \end{aligned} \tag{2.13}$$

We can get similar result for the spiked model with rank r with the same idea, special values of Schur polynomials like (2.6) and (2.7) and more laborious work:

Proposition 2.1. *In a rank k spiked model, let r population eigenvalues be $1 + \alpha_1, \dots, 1 + \alpha_r$ and other $N - r$ population eigenvalues be 1. Some α_j 's may not be distinct, but they assume values $a_1 < \dots < a_s$ with r_1 of them being a_1, \dots, r_s of them being a_s and $\sum_{j=1}^l r_j = r$. If we take M measurements, the joint p.d.f. of sample*

eigenvalues is

$$P(\lambda) = \frac{1}{C} V(\lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-r-1} & \lambda_2^{N-r-1} & \dots & \lambda_N^{N-r-1} \\ p_1 \left(\frac{a_1}{1+a_1} M \lambda_1 \right) & p_1 \left(\frac{a_1}{1+a_1} M \lambda_2 \right) & \dots & p_1 \left(\frac{a_1}{1+a_1} M \lambda_N \right) \\ \vdots & \vdots & \dots & \vdots \\ p_{r_1} \left(\frac{a_1}{1+a_1} M \lambda_1 \right) & p_{r_1} \left(\frac{a_1}{1+a_1} M \lambda_2 \right) & \dots & p_{r_1} \left(\frac{a_1}{1+a_1} M \lambda_N \right) \\ \vdots & \vdots & \dots & \vdots \\ p_{r_s} \left(\frac{a_s}{1+a_s} M \lambda_1 \right) & p_{r_s} \left(\frac{a_s}{1+a_s} M \lambda_2 \right) & \dots & p_{r_s} \left(\frac{a_s}{1+a_s} M \lambda_N \right) \end{vmatrix} \prod_{j=1}^N \lambda_j^{M-N} e^{-M \lambda_j}, \quad (2.14)$$

where the determinant is similar to a Vandermonde, the only difference being in the last r rows: If $\sum_{i=1}^{s'-1} r_i < r' \leq \sum_{i=1}^{s'} r_i$ and $r' - \sum_{i=1}^{s'-1} r_i = t'$, then in the $N - r + r'$ row, the entries are of the form $p_{t'} \left(\frac{a_{s'}}{1+a_{s'}} M \lambda_j \right)$, where

$$p_j(x) = x^{j-1} e^x. \quad (2.15)$$

Later in this thesis we will denote the second determinant in (2.14) by $\tilde{V}(\lambda)$. We will not give the inductive proof of proposition 2.1 like the formula (2.13) for the $r = 1$ case, because it can be proven much easier by the Harish-Chandra-Itzykson-Zuber (HCIZ) formula, see e.g. [21]:

Lemma 2.1 (HCIZ formula). *Given two $N \times N$ Hermitian matrices X and Y , each having distinct eigenvalues x_1, \dots, x_N and y_1, \dots, y_N , we have*

$$\int_{U(N)} e^{\text{Tr}(XUYU^{-1})} dU = \frac{1}{C} \frac{\det(e^{x_i y_j})_{1 \leq i, j \leq N}}{V(x)V(y)}. \quad (2.16)$$

Sketch of proof of proposition 2.1 by HCIZ formula. By the HCIZ formula, we can simplify the joint p.d.f. (1.35) of generic Wishart ensemble (i.e., all population eigenvalues are distinct) as

$$\begin{aligned} P(\lambda) &= \frac{1}{C} V(\lambda) \det(e^{-M\sigma_i^{-1}\lambda_j})_{1 \leq i, j \leq N} \prod_{j=1}^N \lambda_j^{M-N} \\ &= \frac{1}{C(l_1, \dots, l_N)} V(\lambda) \det(e^{Ml_i\lambda_j})_{1 \leq i, j \leq N} \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j}, \end{aligned} \quad (2.17)$$

where we denote $l_i = 1 - \sigma_i^{-1}$, and we regard $\{l_i\}$ as a set of parameters, so that the constant C is a function of l_i 's.

If σ_i 's are not distinct, which is equivalent to that l_i 's are not distinct, then $\det(e^{Ml_i\lambda_j})_{1 \leq i, j \leq N} = 0$, and heuristically, to make $P(\lambda)$ a p.d.f. whose total probability is 1, $C(l_1, \dots, l_N)$ must be also 0. In that case, formula (2.17) becomes $\frac{0}{0}$, and we can apply l'Hôpital's rule to these multiple l_j 's to get a reasonable formula.

For example, if $l_1 = \dots = l_{N-r} = 0$ and l_{N-r+1}, \dots, l_N are distinct numbers other than 0, then we have

$$\begin{aligned} P(\lambda) &= \frac{\mathcal{D} \det(e^{Ml_i\lambda_j})_{1 \leq i, j \leq N}}{\mathcal{D} C(l_1, \dots, l_N)} \Big|_{l_1 = \dots = l_{N-r} = 0} V(\lambda) \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j} \\ &= \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{N-r-1} & \lambda_2^{N-r-1} & \dots & \lambda_N^{N-r-1} \\ e^{Ml_{N-r+1}\lambda_1} & e^{Ml_{N-r+1}\lambda_2} & \dots & e^{Ml_{N-r+1}\lambda_N} \\ \vdots & \vdots & \dots & \vdots \\ e^{Ml_N\lambda_1} & e^{Ml_N\lambda_2} & \dots & e^{Ml_N\lambda_N} \end{vmatrix}}{\mathcal{D} C(l_1, \dots, l_N)} V(\lambda) \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j}, \end{aligned} \quad (2.18)$$

where \mathcal{D} is the differential operator

$$\mathcal{D} = \frac{\partial^{(N-r)(N-r-1)/2}}{\partial l_2 \partial l_3^2 \dots \partial l_{N-r}^{N-r-1}}. \quad (2.19)$$

The formula (2.18) agrees with (2.14), if we regard $\mathcal{D}C(l_1, \dots, l_N)$ as the constant. For the case that l_{N-r+1}, \dots, l_N are not distinct, we apply l'Hôpital's rule also to them and still get (2.14). \square

2.2 Multiple orthogonal Laguerre polynomials

Our goal is to find the probability that the largest sample eigenvalue is less than a certain value, which is

$$\begin{aligned} \mathbb{P}(\max(\lambda) < T) &= \int_0^T \dots \int_0^T P(\lambda) d\lambda_1 \dots \lambda_N \\ &= \frac{1}{C} \int_0^T \dots \int_0^T V(\lambda) \tilde{V}(\lambda) \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j} d\lambda_1 \dots \lambda_N. \end{aligned} \quad (2.20)$$

To evaluate the integral of determinants (2.20), we change it into a determinant of integrals [8]:

Lemma 2.2 (de Bruijn's). *For any two sets of functions f_1, \dots, f_N and g_1, \dots, g_N defined over $[a, b]$, we have*

$$\begin{aligned} \int_a^b \dots \int_a^b \begin{vmatrix} f_1(x_1) & \dots & f_1(x_N) \\ \vdots & \dots & \vdots \\ f_N(x_1) & \dots & f_N(x_N) \end{vmatrix} \begin{vmatrix} g_1(x_1) & \dots & g_1(x_N) \\ \vdots & \dots & \vdots \\ g_N(x_1) & \dots & g_N(x_N) \end{vmatrix} dx_1 \dots dx_N \\ = n! \det \left(\int_a^b f_i(x) g_j(x) dx \right)_{1 \leq i, j \leq N}. \end{aligned} \quad (2.21)$$

The verification of the integral formula is straightforward.

Before the application of lemma 2.2, we are going to do some preparation to (2.14). Let $\varphi_0, \dots, \varphi_{N-1}$ be polynomials of degree $0, \dots, N-1$ respectively, i.e., $\varphi_j(x)$ is a linear combination of functions $1, x, \dots, x^j$, which are functions defining the first $j+1$ rows in the Vandermonde matrix $V(\lambda)$. Similarly, let $\tilde{\varphi}_j$ ($0 \leq j \leq N-1$) be a linear combination of the $j+1$ functions which define the first $j+1$ rows in $\tilde{V}(\lambda)$, so that $\tilde{\varphi}_0, \dots, \tilde{\varphi}_{N-r-1}$ are polynomials but $\tilde{\varphi}_{N-r}, \dots, \tilde{\varphi}_{N-1}$ are not. We require that for $0 \leq i, j \leq N-1$,

$$\int_0^\infty \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx = \delta_{ij}. \quad (2.22)$$

These orthogonal conditions cannot determine φ_j and $\tilde{\varphi}_j$ uniquely, since we can multiply a constant C to φ_j and $1/C$ to $\tilde{\varphi}_j$. For $0 \leq j \leq N-r-1$, the conditions for φ_j and $\tilde{\varphi}_j$ are symmetric, so $\tilde{\varphi}_j = C\varphi_j$, a constant multiple of φ_j , and (2.22) gives

$$\int_0^\infty C\varphi_i(x) \varphi_j(x) x^{M-N} e^{-Mx} dx = \delta_{ij}, \quad (2.23)$$

so that we can choose for arbitrary C_j

$$\varphi_j(x) = C_j L_j^{(M-N)}(Mx), \quad (2.24)$$

$$\tilde{\varphi}_j(x) = \frac{j! M^{M-N+1}}{(M-N+j)! C_j} L_j^{(M-N)}(Mx), \quad (2.25)$$

since

$$\int_0^\infty L_i^{(M-N)}(x) L_j^{(M-N)}(x) x^{M-N} e^{-x} dx = \delta_{ij} \frac{(M-N+j)!}{j!}. \quad (2.26)$$

For notational simplicity, we denote the inner product of two functions $f(x)$ and $g(x)$ on $[0, \infty)$ by $\langle \cdot, \cdot \rangle_2$:

$$\langle f(x), g(x) \rangle_2 = \int_0^\infty f(x) g(x) x^{M-N} e^{-Mx} dx. \quad (2.27)$$

For $j \geq N - r$, we index φ_j and $\tilde{\varphi}_j$ by a triple (r', s', t') , which appears in the statement of proposition 2.1, and is defined as

$$r' \in \{1, \dots, r\}, \quad r' = \left(\sum_{j=1}^{s'-1} r_j \right) + t', \quad 1 \leq t' \leq r_{s'}. \quad (2.28)$$

It is Bleher and Kuijlaars' observation that [7]

Proposition 2.2. *Given (r', s', t') defined as (2.28), we have*

$$\begin{aligned} \varphi_{N-r+r'-1}(x) &= \frac{C_{r'}}{x^{M-N} 2\pi i} \\ &\oint_{\Sigma} e^{Mxz} \frac{(z-1)^{N-r}}{z^{M-r+r'}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz, \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} \tilde{\varphi}_{N-r+r'-1}(x) &= \frac{Me^{Mx}}{(1+a_{s'})C_{r'} 2\pi i} \\ &\oint_{\Gamma} e^{-Mxz} \frac{z^{M-r+r'-1}}{(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz, \end{aligned} \quad (2.30)$$

where $C_{r'}$ is an arbitrary constant, Σ is a contour around $z = 0$ and is to the left of the points $z = 1$ and $z = \frac{1}{1+a_j}$ ($j = 1, \dots, s'$), and Γ is a contour containing the points $z = 1$ and $z = \frac{1}{1+a_j}$ ($j = 1, \dots, s'$), and is to the right of $z = 0$.

Sketch of proof. By the residue theorem, we can check that $\varphi_{N-r+r'-1}(x)$ defined in (2.29) is a polynomial of degree $N - r + r' - 1$, and $\tilde{\varphi}_{N-r+r'-1}(x)$ defined in (2.30) is also in the correct form: For any r' ,

$$\begin{aligned} \tilde{\varphi}_{N-r+r'-1}(x) &= c_0 1 + c_1 x + \dots + c_{N-r-1} x^{N-r-1} \\ &\quad + c_{N-r} p_1 \left(\frac{a_1}{1+a_1} Mx \right) + \dots + c_{N-r+r'-1} p_{t'} \left(\frac{a_{s'}}{1+a_{s'}} Mx \right), \end{aligned} \quad (2.31)$$

where the j -th term is a constant c_j times the function that defines the j -th row in the determinant in (2.14). Then we can check that for $k = 0, \dots, N - r - 1$,

$$\begin{aligned}
& \langle \varphi_{N-r+r'-1}(x), x^k \rangle_2 \\
&= \int_0^\infty x^k \frac{C'_{r'}}{2\pi i} \oint_\Sigma e^{Mx(z-1)} \frac{(z-1)^{N-r}}{z^{M-r+r'}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz dx \\
&= \frac{C'_{r'}}{2\pi i} \oint_\Sigma \left(\int_0^\infty x^k e^{Mx(z-1)} dx \right) \frac{(z-1)^{N-r}}{z^{M-r+r'}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz \\
&= \frac{C'_{r'}}{2\pi i} \oint_\Sigma \frac{k!}{(M(1-z))^{k+1}} \frac{(z-1)^{N-r}}{z^{M-r+r'}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz \\
&= \frac{k! C'_{r'}}{(-M)^{k+1} 2\pi i} \oint_\Sigma (z-1)^{N-r-k-1} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} \frac{dz}{z^{M-r+r'}},
\end{aligned} \tag{2.32}$$

which is 0 by the residue theorem. Similarly, we can prove for $j \leq s' - 1$, $i \leq r_j$ or $j = s'$, $i < t'$, using (2.15), that

$$\left\langle \varphi_{N-r+r'-1}(x), p_i \left(\frac{a_j}{1+a_j} Mx \right) \right\rangle_2 = 0, \tag{2.33}$$

so that by linearity, for $0 \leq j < N - r + r' - 1$, using (2.31), we have

$$\langle \varphi_{N-r+r'-1}(x), \tilde{\varphi}_j(x) \rangle_2 = 0, \tag{2.34}$$

In the same way, we can check that for $0 \leq j < N - r + r' - 1$,

$$\langle x^j, \tilde{\varphi}_{N-r+r'-1}(x) \rangle_2 = 0, \tag{2.35}$$

$$\langle \varphi_j(x), \tilde{\varphi}_{N-r+r'-1}(x) \rangle_2 = 0. \tag{2.36}$$

We can compute the leading term of $\varphi_{N-r+r'-1}(x)$, i.e., the $x^{N-r+r'-1}$ term, from

(2.29), which is

$$\begin{aligned} & \frac{C_{r'}}{x^{M-N} 2\pi i} \oint_{\Sigma} \frac{(Mxz)^{M-r+r'-1}}{(M-r+r'-1)!} \frac{(z-1)^{N-r}}{z^{M-r+r'}} \\ & \quad \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz = \\ & (-1)^{N-r+r'-1} \frac{M^{M-r+r'-1} C_{r'}}{(M-r+r'-1)!} \left(\prod_{j=1}^{s'-1} \frac{1}{(1+a_j)^{r_j}} \right) \frac{1}{(1+a_{s'})^{t'-1}} x^{N-r+r'-1}. \end{aligned} \quad (2.37)$$

Therefore by (2.35) and (2.37), if we denote $\tilde{C}_{r'} = \left(\prod_{j=1}^{s'-1} \frac{1}{(1+a_j)^{r_j}} \right) \frac{1}{(1+a_{s'})^{t'}}$,

$$\begin{aligned} & \langle \varphi_{N-r+r'-1}(x), \tilde{\varphi}_{N-r+r'-1}(x) \rangle_2 \\ &= \left\langle (-1)^{N-r+r'-1} \frac{(-M)^{M-r+r'-1} C_{r'}}{(M-r+r'-1)!} \tilde{C}_{r'-1} x^{N-r+r'-1}, \tilde{\varphi}_{N-r+r'-1}(x) \right\rangle_2 \\ &= (-1)^{N-r+r'-1} \frac{M^{M-r+r'}}{(M-r+r'-1)!} \frac{\tilde{C}_{r'-1}}{1+a_{s'}} \int_0^\infty x^{M-r+r'-1} \\ & \quad \frac{1}{2\pi i} \oint_{\Gamma} e^{-Mxz} \frac{z^{M-r+r'-1}}{(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz dx \\ &= (-1)^{N-r+r'-1} \frac{M^{M-r+r'} \tilde{C}_{r'}}{(M-r+r'-1)!} \frac{1}{2\pi i} \oint_{\Gamma} \left(\int_0^\infty x^{M-r+r'-1} e^{-Mxz} dx \right) \\ & \quad \frac{z^{M-r+r'-1}}{(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz \\ &= (-1)^{N-r+r'-1} \tilde{C}_{r'} \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{z(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz \\ &= (-1)^{N-r+r'-1} \tilde{C}_{r'} \frac{1}{2\pi i} \left[\oint_{\Gamma'} \frac{1}{z(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz \right. \\ & \quad \left. - \oint_{\Gamma''} \frac{1}{z(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz \right], \end{aligned} \quad (2.38)$$

where Γ' is a contour including 0, 1 and $\frac{1}{1+a_j}$ ($j = 1, \dots, s$), and Γ'' is a contour

including 0 and excluding 1 and $\frac{1}{1+a_j}$. We can deform Γ' to infinity to see that

$$\frac{1}{2\pi i} \oint_{\Gamma'} \frac{1}{z(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz = 0, \quad (2.39)$$

and by the residue theorem see that

$$\frac{1}{2\pi i} \oint_{\Gamma''} \frac{1}{z(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz = \frac{(-1)^{N-r+r'}}{\tilde{C}_{r'}}, \quad (2.40)$$

and we check that $\langle \varphi_{N-r+r'-1}(x), \tilde{\varphi}_{N-r+r'-1}(x) \rangle_2 = 1$. \square

Although the integral formulas for $\varphi_{N-r+r'-1}(x)$ and $\tilde{\varphi}_{N-r+r'-1}(x)$ seems strange, if we compare $\tilde{\varphi}_{N-r+r'-1}(x)$ with the integral representation of Laguerre polynomials [27]

$$L_j^{(M-N)}(Mx) = \frac{e^{Mx}}{2\pi i} \oint_{\Gamma} e^{-Mxz} \frac{z^{M-N+j}}{(z-1)^{j+1}} dz, \quad (2.41)$$

and $\varphi_{N-r+r'-1}(x)$ with another integral representation of Laguerre polynomials

$$L_j^{(M-N)}(Mx) = \frac{(M-N+j)!}{j!M^{M-N}} \frac{1}{x^{M-N}2\pi i} \oint_{\Sigma} e^{Mxz} \frac{(z-1)^j}{z^{M-N+j+1}} dz, \quad (2.42)$$

and find that they are variations of Laguerre polynomials. The $\varphi_{N-r+r'-1}(x)$'s are called multiple Laguerre polynomials of type II, and the $\tilde{\varphi}_{N-r+r'-1}(x)$'s are equivalent to—although they are not polynomials—multiple Laguerre polynomials of type I, by Bleher and Kuijlaars [7].

We know that

$$V(\lambda) = \frac{1}{C} \begin{vmatrix} \varphi_0(\lambda_1) & \cdots & \varphi_0(\lambda_N) \\ \vdots & \cdots & \vdots \\ \varphi_{N-1}(\lambda_1) & \cdots & \varphi_{N-1}(\lambda_N) \end{vmatrix}, \quad (2.43)$$

and

$$\tilde{V}(\lambda) = \frac{1}{C} \begin{vmatrix} \tilde{\varphi}_0(\lambda_1) & \dots & \tilde{\varphi}_0(\lambda_N) \\ \vdots & \dots & \vdots \\ \tilde{\varphi}_{N-1}(\lambda_1) & \dots & \tilde{\varphi}_{N-1}(\lambda_N) \end{vmatrix}, \quad (2.44)$$

so that we can write the formula (2.14) for the joint p.d.f. as

$$P(\lambda) = \frac{1}{C} \begin{vmatrix} \varphi_0(\lambda_1) & \dots & \varphi_0(\lambda_N) \\ \vdots & \dots & \vdots \\ \varphi_{N-1}(\lambda_1) & \dots & \varphi_{N-1}(\lambda_N) \end{vmatrix} \begin{vmatrix} \tilde{\varphi}_0(\lambda_1) & \dots & \tilde{\varphi}_0(\lambda_N) \\ \vdots & \dots & \vdots \\ \tilde{\varphi}_{N-1}(\lambda_1) & \dots & \tilde{\varphi}_{N-1}(\lambda_N) \end{vmatrix} \prod_{j=1}^N \lambda_j^{M-N} e^{-M\lambda_j}, \quad (2.45)$$

and by formulas (2.20) and (2.21), we have

$$\mathbb{P}(\max(\lambda) < T) = \frac{1}{C} \det \left(\int_0^T \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx \right)_{0 \leq i, j \leq N-1}, \quad (2.46)$$

where we choose the f_i and g_j in (2.21) to be

$$f_i(x) = \varphi_{i-1}(x) x^{\frac{M-N}{2}} e^{-\frac{M}{2}x}, \quad (2.47)$$

$$g_j(x) = \tilde{\varphi}_{j-1}(x) x^{\frac{M-N}{2}} e^{-\frac{M}{2}x}. \quad (2.48)$$

By (2.22), we have

$$\int_0^T \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx = \delta_{ij} - \int_T^\infty \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx, \quad (2.49)$$

and so

$$\det \left(\int_0^T \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx \right)_{0 \leq i, j \leq N-1} = \det \left(I_N - \left(\int_T^\infty \varphi_i(x) \tilde{\varphi}_j(x) x^{M-N} e^{-Mx} dx \right)_{0 \leq i, j \leq N-1} \right). \quad (2.50)$$

To evaluate the determinant in (2.50), we apply the formula

$$\det \left(I_N - \left(\int_a^b f_i(x) g_j(x) dx \right)_{1 \leq i, j \leq N} \right) = \det \left(I - \chi_{[a, b]}(x) \left(\sum_{j=1}^N f_j(x) g_j(y) \right) \chi_{[a, b]}(y) \right), \quad (2.51)$$

where $\chi_{[a, b]}$ is the indicator function for the interval $[a, b]$, and the latter determinant is the Fredholm determinant for integral operator. Since the integral kernel on the left hand side of (2.51) is of finite rank, we can check the identity directly. Consider the parametrized determinant

$$\det \left(I_N - t \left(\int_a^b f_i(x) g_j(x) dx \right)_{1 \leq i, j \leq N} \right) = 1 + c_1 t + c_2 t^2 + \cdots + c_N t^N, \quad (2.52)$$

we can compute the coefficients degree by degree ($K(x, y) = \sum_{j=1}^N f_j(x) g_j(y)$):

$$c_1 = - \sum_{j=1}^N \int_a^b f_j(x) g_j(x) dx = - \int_a^b K(x_1, x_1) dx_1, \quad (2.53)$$

$$\begin{aligned} c_2 &= \sum_{i, j=1}^N \left(\int_a^b f_i(x) g_i(x) dx \int_a^b f_j(x) g_j(x) dx - \int_a^b f_i(x) g_j(x) dx \int_a^b f_j(x) g_i(x) dx \right) \\ &= \frac{1}{2!} \int_a^b \int_a^b \begin{vmatrix} K(x_1, x_1) & K(x_1, x_2) \\ K(x_2, x_1) & K(x_2, x_2) \end{vmatrix} dx_1 dx_2, \end{aligned} \quad (2.54)$$

.....

and we find the right-hand side of (2.52) is the same as the right-hand side of (1.57) for $t = 1$.

Now using (2.46), (2.50) and (2.51) we get the Fredholm determinantal formula

for the probability that the largest eigenvalue is less than T : ($\chi(x)$ means $\chi_{[T,\infty)}(x)$)

$$\mathbb{P}(\max(\lambda) < T) = \frac{1}{C} \det \left(I - \chi(x) \left(\sum_0^{N-1} \varphi_j(x) \tilde{\varphi}_j(y) x^{\frac{M-N}{2}} y^{\frac{M-N}{2}} e^{-\frac{M}{2}(x+y)} \right) \chi(x) \right). \quad (2.55)$$

Furthermore, we can determine the constant $C = 1$, since letting $T \rightarrow \infty$, $\mathbb{P}(\max(\lambda) < T) \rightarrow 1$ and the Fredholm determinant on the right hand side also approaches 1. To undertake real calculations, we need methods to evaluate of the Fredholm determinant other than the formula (1.57). First, we have the result on taking the limit, see, e.g. [17]:

Proposition 2.3. *If a series of integral operators K_n approaches K in trace norm, then*

$$\lim_{n \rightarrow \infty} \det(I - K_n) = \det(I - K). \quad (2.56)$$

Since as $M \rightarrow \infty$, we expect that the fluctuation scale of the largest eigenvalue shrinks, depending on M . We will consider for any M the probability $\mathbb{P}(\max(\lambda) < p + qT)$, where p and q may depend on M . (It turns out that p is a constant.) If we denote

$$K_{2a}(x, y) = \sum_{j=0}^{N-r-1} \varphi_j(x) \tilde{\varphi}_j(y) x^{\frac{M-N}{2}} y^{\frac{M-N}{2}} e^{-\frac{M}{2}(x+y)}, \quad (2.57)$$

$$K_{2b}(x, y) = \sum_{j=N-r}^{N-1} \varphi_j(x) \tilde{\varphi}_j(y) x^{\frac{M-N}{2}} y^{\frac{M-N}{2}} e^{-\frac{M}{2}(x+y)}, \quad (2.58)$$

$$K_2(x, y) = K_{2a}(x, y) + K_{2b}(x, y), \quad (2.59)$$

and

$$\tilde{K}_*(\xi, \eta) = qK_*(p + q\xi, p + q\eta), \quad (2.60)$$

where $*$ stands for 2, $2a$ or $2b$, then we have the determinantal formula

$$\begin{aligned}\mathbb{P}(\max(\lambda) < p + qT) &= \det \left(I - \chi_{[p+qT, \infty)}(x) K_2(x, y) \chi_{[p+qT, \infty)}(y) \right) \\ &= \det \left(I - \chi(\xi) \tilde{K}_2(\xi, \eta) \chi(\eta) \right).\end{aligned}\quad (2.61)$$

$K_{2a}(x, y)$, which is composed of Laguerre polynomials, is the kernel for the LUE, and we can write it in an integral form:

Theorem 2.1.

$$\begin{aligned}K_{2a}(x, y) &= -M^2 \frac{y^{\frac{M-N}{2}} e^{\frac{M}{2}y}}{x^{\frac{M-N}{2}} e^{\frac{M}{2}x}} \int_0^\infty \left(\frac{1}{2\pi i} \oint_\Gamma e^{-M(y+t)z} \frac{z^{M-r}}{(z-1)^{N-r}} dz \right) \\ &\quad \left(\frac{1}{2\pi i} \oint_\Sigma e^{M(x+t)w} \frac{(w-1)^{N-r}}{w^{M-r}} dw \right) dt.\end{aligned}\quad (2.62)$$

Proof. Because of (2.24), (2.25), (2.41), (2.42) and (2.57), we have the integral representation

$$\begin{aligned}&K_{2a}(x, y) \\ &= \sum_{j=0}^{N-r-1} \frac{j!}{(M-N+j)!} M^{M-N+1} L_j^{(M-N)}(Mx) L_j^{(M-N)}(My) x^{\frac{M-N}{2}} y^{\frac{M-N}{2}} e^{-\frac{M}{2}(x+y)} \\ &= \frac{M}{(2\pi i)^2} \frac{y^{\frac{M-N}{2}} e^{\frac{M}{2}y}}{x^{\frac{M-N}{2}} e^{\frac{M}{2}x}} \sum_{j=0}^{N-r-1} \oint_\Gamma dz \oint_\Sigma dw e^{-Myz} \frac{z^{M-N+j}}{(z-1)^{j+1}} e^{Mxw} \frac{(w-1)^j}{w^{M-N+j+1}}.\end{aligned}\quad (2.63)$$

We can write the sum of integrands in (2.63) as

$$\begin{aligned}
& \sum_{j=0}^{N-r-1} e^{Mxz} \frac{z^{M-N+j}}{(z-1)^{j+1}} e^{-Myw} \frac{(w-1)^j}{w^{M-N+j+1}} \\
&= e^{Mxz} e^{-Myw} \frac{z^{M-N}}{w^{M-N}} \frac{1}{(z-1)w} \sum_{j=0}^{N-r-1} \left(\frac{z(w-1)}{(z-1)w} \right)^j \\
&= e^{Mxz} e^{-Myw} \frac{z^{M-N}}{w^{M-N}} \frac{1}{(z-1)w} \frac{1 - \left(\frac{z(w-1)}{(z-1)w} \right)^{N-r}}{1 - \frac{z(w-1)}{(z-1)w}} \\
&= \frac{1}{z-w} e^{Mxz} z^{M-N} e^{-Myw} \frac{1}{w^{M-N}} \\
&\quad - \frac{1}{z-w} e^{Mxz} \frac{z^{M-r}}{(z-1)^{N-r}} e^{-Myw} \frac{(w-1)^{N-r}}{w^{M-r}}.
\end{aligned} \tag{2.64}$$

By residue theorem, since Γ and Σ are disjoint, and for the variable z , the pole $z = w$ is outside of Γ ,

$$\oint_{\Gamma} dz \oint_{\Sigma} dw \frac{1}{z-w} e^{Mxz} z^{M-N} e^{-Myw} \frac{1}{w^{M-N}} = 0. \tag{2.65}$$

On the other side, since $\Re(w-z)$ is always less than 0,

$$\frac{1}{z-w} = M \int_0^{\infty} e^{tM(w-z)} dt, \tag{2.66}$$

so that we have

$$\begin{aligned}
& \frac{M}{(2\pi i)^2} \oint_{\Gamma} dz \oint_{\Sigma} dw \frac{1}{z-w} e^{-Myz} \frac{z^{M-r}}{(z-1)^{N-r}} e^{Mxw} \frac{(w-1)^{N-r}}{w^{M-r}} \\
&= \frac{M^2}{(2\pi i)^2} \oint_{\Gamma} dz \oint_{\Sigma} dw \int_0^{\infty} e^{-M(y+t)z} \frac{z^{M-r}}{(z-1)^{N-r}} e^{M(x+t)w} \frac{(w-1)^{N-r}}{w^{M-r}} \\
&= M^2 \int_0^{\infty} \left(\frac{1}{2\pi i} \oint_{\Gamma} e^{-M(y+t)z} \frac{z^{M-r}}{(z-1)^{N-r}} dz \right) \left(\frac{1}{2\pi i} \oint_{\Sigma} e^{M(x+t)w} \frac{(w-1)^{N-r}}{w^{M-r}} dw \right) dt.
\end{aligned} \tag{2.67}$$

Put (2.63)–(2.67) together, we get the result. \square

Now $K_{2a}(x, y)$ ($\tilde{K}_{2a}(\xi, \eta)$) is expressed by two functions and $K_{2b}(x, y)$ ($\tilde{K}_{2b}(\xi, \eta)$) is a finite rank operator. To undertake the asymptotic analysis, we need two propositions on the convergence in trace norm:

Proposition 2.4. *Let $\{f_{j,n}\}$ and $\{g_{j,n}\}$ ($1 \leq j \leq m$, $1 \leq n < \infty$) be $2m$ series of functions on $[T, \infty)$ and $f_{j,n} \rightarrow f_j$, $g_{j,n} \rightarrow g_j$ in L^2 norm on $[T, \infty)$. We have the convergence in trace norm of operators $(K_n(x, y) = \sum_{j=1}^m f_{j,n}(x)g_{j,n}(x))$, $K(x, y) = \sum_{j=1}^m f_j(x)g_j(x)$:*

$$\lim_{n \rightarrow \infty} \chi(x)K_n(x, y)\chi(y) = \chi(x)K(x, y)\chi(y). \quad (2.68)$$

Proposition 2.5. *Let $f_n(x)$, $g_n(y)$ be two series of functions on $[T, \infty)$ and in L^2 norm we have*

$$\lim_{n \rightarrow \infty} \|(f_n(x) - f(x))(x - T)\|_{L^2([T, \infty))} = 0, \quad (2.69)$$

$$\lim_{n \rightarrow \infty} \|(g_n(y) - g(y))(y - T)\|_{L^2([T, \infty))} = 0, \quad (2.70)$$

then we have the convergence in trace norm of operators $(K_n(x, y) = \int_0^\infty f_n(x + t)g_n(y + t)dt)$, $K(x, y) = \int_0^\infty f(x + t)g(y + t)dt$:

$$\lim_{n \rightarrow \infty} \chi(x)K_n(x, y)\chi(y) = \chi(x)K(x, y)\chi(y). \quad (2.71)$$

We can verify proposition 2.4 by the definition of trace norm. Proposition 2.5 is essentially a fact of trace norm convergence [17]: If I_n , J_n are Hilbert-Schmidt operators and $I_n \rightarrow I$, $J_n \rightarrow J$ in the Hilbert-Schmidt norm, then the product $I_n J_n$ and IJ are trace class operators and $I_n J_n \rightarrow IJ$ in trace norm. Here we take I_n as the integral operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with the kernel $\chi(x)f_n(x + y)\chi_{[0, \infty)}(y)$, and J_n also an integral operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with the kernel $\chi_{[0, \infty)}(x)g_n(x + y)\chi(y)$.

For an integral operator, the Hilbert-Schmidt norm is equivalent to the L^2 norm of its kernel as a 2 variable function. In our special case, the convergence results of $L^2(\mathbb{R}^2)$ is equivalent to the convergence of $L^2([T, \infty))$ in (2.69) and (2.70), due to the Fubini's theorem.

To apply propositions 2.4 and 2.5, we sometimes need to conjugate the integral operator by a weight function:

Proposition 2.6. *We have*

$$\det(1 - K(x, y)) = \det(1 - f(x)K(x, y)f^{-1}(y)), \quad (2.72)$$

for any function f which make the right hand side of (2.72) well defined.

Proof. A direct application of the definition of Fredholm determinant (1.57). \square

If we take $f(x) = x^{(M-N)/2} e^{\frac{1-\gamma}{2(\gamma+1)}Mx}$, then by (2.29), (2.30) and (2.58) (2.62) we have

$$f(x)K_{2a}(x, y)f^{-1}(y) = -M^2 \int_0^\infty e^{-\frac{\gamma}{\gamma+1}M(x+t)} \psi(x+t) e^{\frac{\gamma}{\gamma+1}M(y+t)} \tilde{\psi}(y+t) dt, \quad (2.73)$$

$$f(x)K_{2b}(x, y)f^{-1}(y) = M \sum_{r'=1}^r \frac{1}{1+a_{s'}} e^{-\frac{\gamma}{\gamma+1}M(x+t)} \psi_{r'}(x) e^{\frac{\gamma}{\gamma+1}M(y+t)} \tilde{\psi}_{r'}(y), \quad (2.74)$$

where

$$\psi(x) = \frac{1}{2\pi i} \oint_{\Sigma} e^{Mxz} \frac{(z-1)^{N-r}}{z^{M-r}} dz, \quad (2.75)$$

$$\tilde{\psi}(y) = \frac{1}{2\pi i} \oint_{\Gamma} e^{-Myz} \frac{z^{M-r}}{(z-1)^{N-r}} dz, \quad (2.76)$$

$$\psi_{r'}(x) = \frac{1}{2\pi i} \oint_{\Sigma} e^{Mxz} \frac{(z-1)^{N-r}}{z^{M-r+r'}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz, \quad (2.77)$$

$$\tilde{\psi}_{r'}(y) = \frac{1}{2\pi i} \oint_{\Gamma} e^{-Myz} \frac{z^{M-r+r'-1}}{(z-1)^{N-r}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz. \quad (2.78)$$

2.3 Proof of theorem 1.1

We give proofs of all the three parts separately. In this section, we let $x = p + q\xi$ and $y = q + q\eta$.

2.3.1 The $-1 < a_s < \gamma^{-1}$ part

In this case, we choose $p = (1 + \gamma^{-1})^2$ and $q = \frac{(\gamma+1)^{4/3}}{\gamma M^{2/3}}$, and by (2.60), write (2.73) as

$$f(x) \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) = - \int_0^\infty \Psi(\xi + t) \tilde{\Psi}(\eta + t) dt, \quad (2.79)$$

where

$$\Psi(\xi) = \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi(p + q\xi), \quad (2.80)$$

$$\tilde{\Psi}(\eta) = \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} My} \tilde{\psi}(p + q\eta). \quad (2.81)$$

Since (5.51) and (5.71) give the L^2 convergence results

$$\lim_{M \rightarrow \infty} \|(\Psi(\xi) - (-\gamma)^r \text{Ai}(\xi))(\xi - T)\|_{L^2([T, \infty))} = 0, \quad (2.82)$$

$$\lim_{M \rightarrow \infty} \left\| (\tilde{\Psi}(\eta) - (-1)^{r-1} \gamma^{-r} \text{Ai}(\eta))(\eta - T) \right\|_{L^2([T, \infty))} = 0, \quad (2.83)$$

we get the convergence in trace norm by proposition 2.5

$$\begin{aligned} & \lim_{M \rightarrow \infty} \chi(\xi) f(x) \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) \chi(\eta) \\ &= -\chi(\xi) \int_0^\infty (-\gamma)^r \text{Ai}(\xi + t) (-1)^{r-1} \gamma^{-r} \text{Ai}(\eta + t) dt \chi(\eta) \\ &= \chi(\xi) K_{\text{Airy}}(\xi, \eta) \chi(\eta). \end{aligned} \quad (2.84)$$

Similarly, for any $r' = 1, \dots, r$, by (2.74) and (2.60),

$$f(x) \tilde{K}_{2b}(\xi, \eta) f^{-1}(y) = \frac{\gamma}{(\gamma + 1)^{4/3} M^{1/3}} \sum_{r'=1}^r \frac{1}{1 + a_{s'}} \Psi_{r'}(\xi) \tilde{\Psi}_{r'}(\eta), \quad (2.85)$$

where

$$\Psi_{r'}(\xi) = \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{\gamma^M}{(\gamma + 1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi_{r'}(p + q\xi), \quad (2.86)$$

$$\tilde{\Psi}_{r'}(\eta) = \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{(\gamma + 1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} My} \psi_{r'}(p + q\eta). \quad (2.87)$$

Because (5.50) and (5.70) imply the L^2 convergence results ($\bar{C}_{r'}$ is defined in (5.45))

$$\lim_{M \rightarrow \infty} \Psi_{r'}(\xi) \chi(\xi) = (-1)^r \gamma^{r-r'} (\gamma + 1)^{r'} \bar{C}_{r'-1} \text{Ai}(\xi) \chi(\xi), \quad (2.88)$$

$$\lim_{M \rightarrow \infty} \tilde{\Psi}_{r'}(\eta) \chi(\eta) = \frac{(-1)^r (1 + \gamma^{-1})}{\gamma^{r-r'} (\gamma + 1)^{r'} \bar{C}_{r'}} \text{Ai}(\eta) \chi(\eta), \quad (2.89)$$

$$(2.90)$$

we get by proposition 2.4 the boundedness in trace norm (C is a large enough positive

constant)

$$\left\| \chi(\xi) \left(\sum_{r'=1}^r \frac{1}{1+a_{s'}} \Psi_{r'}(\xi) \tilde{\Psi}_{r'}(\eta) \right) \chi(\eta) \right\|_{\text{Tr}} < C, \quad (2.91)$$

so that in trace norm

$$\lim_{M \rightarrow \infty} \chi(\xi) f(x) \tilde{K}_{2b}(\xi, \eta) f^{-1}(y) \chi(\eta) = 0. \quad (2.92)$$

Therefore,

$$\begin{aligned} & \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) \tilde{K}(\xi, \eta) \chi(\eta) \right) \\ &= \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) f(x) \tilde{K}(\xi, \eta) \chi(\eta) f^{-1}(y) \right) \\ &= \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) f(x) \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) \chi(\eta) \right) \\ &= \det \left(1 - \chi(\xi) K_{\text{Airy}}(\xi, \eta) \chi(\eta) \right). \end{aligned} \quad (2.93)$$

2.3.2 The $a_s = \gamma^{-1}$ part

In this case we still choose the same p and q as in the previous case, but we need to conjugate the kernel not only by $f(x)$, but also $e^{\xi/3}$. If we consider

$$f(x) e^{\xi/3} \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) e^{-\eta/3} = - \int_0^\infty e^{(\xi+t)/3} \Psi(\xi+t) e^{-(\eta+t)/3} \tilde{\Psi}(\eta+t) dt, \quad (2.94)$$

and similar to (2.82) and (2.83), (5.51) and (5.71) imply also the L^2 convergence results

$$\lim_{M \rightarrow \infty} \left\| (e^{\xi/3} \Psi(\xi) - (-\gamma)^r e^{\xi/3} \text{Ai}(\xi)) (\xi - T) \right\|_{L^2([T, \infty))} = 0, \quad (2.95)$$

$$\lim_{M \rightarrow \infty} \left\| (e^{-\eta/3} \tilde{\Psi}(\eta) - (-1)^{r-1} \gamma^{-r} e^{-\eta/3} \text{Ai}(\eta)) (\eta - T) \right\|_{L^2([T, \infty))} = 0, \quad (2.96)$$

and we can get the result of trace norm convergence similar to (2.84)

$$\lim_{M \rightarrow \infty} \chi(\xi) f(x) e^{\xi/3} \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) e^{-\eta/3} \chi(\eta) = \chi(\xi) e^{\xi/3} K_{\text{Airy}}(\xi, \eta) e^{-\eta/3} \chi(\eta). \quad (2.97)$$

we can still get by (5.51), (5.71) and proposition 2.5 the trace norm convergence

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \chi(\xi) f(x) e^{\xi/3} \tilde{K}_{2a}(\xi, \eta) f^{-1}(y) e^{-\eta/3} \chi(\eta) \\
&= \chi(\xi) \int_0^\infty e^{(\xi+t)/3} \text{Ai}(\xi+t) e^{-(\eta+t)/3} \text{Ai}(\eta+t) dt \chi(\eta) \\
&= \chi(\xi) e^{\xi/3} K_{\text{Airy}}(\xi, \eta) e^{-\eta/3} \chi(\eta).
\end{aligned} \tag{2.98}$$

We can write the formula (2.74) as

$$\begin{aligned}
f(x) e^{\xi/3} \tilde{K}_{2b}(\xi, \eta) f^{-1}(y) e^{-\eta/3} &= \frac{\gamma}{(\gamma+1)^{4/3} M^{1/3}} \sum_{r'=1}^{r-r_s} \frac{1}{1+a_{s'}} e^{\xi/3} \Psi_{r'}(\xi) e^{-\eta/3} \tilde{\Psi}_{r'}(\eta) \\
&+ \sum_{r'=r-r_s+1}^r \frac{1}{1+\gamma^{-1}} e^{\xi/3} \Psi_{t',r'}(\xi) e^{-\eta/3} \tilde{\Psi}_{t',r'}(\eta), \tag{2.99}
\end{aligned}$$

where

$$\Psi_{t',r'}(\xi) = \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{t'} (-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi_{r'}(p+q\xi), \tag{2.100}$$

$$\tilde{\Psi}_{t',r'}(\eta) = \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{1-t'} (-1)^N \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} My} \psi_{r'}(p+q\eta). \tag{2.101}$$

In the same way of (2.92), we have the convergence result in trace norm

$$\lim_{M \rightarrow \infty} \left\| \frac{\gamma}{(\gamma+1)^{4/3} M^{1/3}} \chi(\xi) \left(\sum_{r'=1}^{r-r_s} \frac{1}{1+a_{s'}} e^{\xi/3} \Psi_{r'}(\xi) e^{-\eta/3} \tilde{\Psi}_{r'}(\eta) \right) \chi(\eta) \right\|_{\text{Tr}} = 0. \tag{2.102}$$

However, if $r' = \left(\sum_j^{s-1} r_j \right) + t'$, $t' = 1, \dots, r_s$, (5.86) and (5.98) yield the convergence in L^2 norm

$$\lim_{M \rightarrow \infty} e^{\xi/3} \Psi(\xi)_{t',r'} \chi(\xi) = (-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-t'} (-1)^{t'-1} e^{\xi/3} t^{(t')}(\xi), \tag{2.103}$$

$$\lim_{M \rightarrow \infty} e^{-\eta/3} \tilde{\Psi}(\eta)_{t',r'} \chi(\eta) = \frac{(-1)^r (1+\gamma^{-1})}{\gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-t'}} (-1)^{t'-1} e^{-\eta/3} s^{(t')}(\xi), \tag{2.104}$$

so that by proposition 2.4 we have the convergence in trace norm

$$\begin{aligned} \lim_{M \rightarrow \infty} \chi(\xi) e^{\xi/3} \left(\sum_{r'=r-r_s+1}^r \frac{1}{1+\gamma^{-1}} e^{\xi/3} \Psi_{t',r'}(\xi) e^{-\eta/3} \tilde{\Psi}_{t',r'}(\eta) \right) e^{-\eta/3} \chi(\eta) = \\ e^{\xi/3} \left(\sum_{j=1}^{r_s} t^{(j)}(\xi) s^{(j)}(\eta) \right) e^{-\eta/3}, \end{aligned} \quad (2.105)$$

which is exactly the trace norm limit of $\chi(\xi) f(x) e^{\xi/3} \tilde{K}_{2b}(\xi, \eta) f^{-1}(y) e^{-\eta/3} \chi(\eta)$.

Therefore,

$$\begin{aligned} & \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) \tilde{K}(\xi, \eta) \chi(\eta) \right) \\ &= \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) f(x) e^{\xi/3} (\tilde{K}_{2a}(\xi, \eta) + \tilde{K}_{2b}(\xi, \eta)) \chi(\eta) f^{-1}(y) e^{-\eta/3} \right) \\ &= \det \left(1 - \chi(\xi) e^{\xi/3} \left(K_{\text{Airy}}(\xi, \eta) + \sum_{j=1}^{r_s} t^{(j)}(\xi) s^{(j)}(\eta) \right) e^{-\eta/3} \chi(\eta) \right) \\ &= \det \left(1 - \chi(\xi) \left(K_{\text{Airy}}(\xi, \eta) + \sum_{j=1}^{r_s} t^{(j)}(\xi) s^{(j)}(\eta) \right) \chi(\eta) \right). \end{aligned} \quad (2.106)$$

2.3.3 The $a_s = a > \gamma^{-1}$ part

In this we choose $p = (1+a) \left(1 + \frac{1}{\gamma^2 a} \right)$ and $q = (1+a) \sqrt{1 - \frac{1}{\gamma^2 a^2} \frac{1}{\sqrt{M}}}$, and conjugate the kernel by $f_a(x) e^{2\xi/3}$, where $f_a(x) = x^{(M-N)/2} e^{\frac{a-1}{2(1+a)} Mx}$, Then by (2.62), (2.75) and (2.76) we have

$$f_a(x) e^{2\xi/3} \tilde{K}_{2a}(\xi, \eta) f_a^{-1}(y) e^{-2\eta/3} = - \int_0^\infty e^{2(\xi+t)/3} \Psi^a(\xi+t) \tilde{\Psi}^a(\eta+t) e^{-2(\eta+t)/3} dt, \quad (2.107)$$

where

$$\Psi^a(\xi) = (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}}} \psi(p + q\xi), \quad (2.108)$$

$$\tilde{\Psi}^a(\eta) = (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a}y}} \tilde{\psi}(p + q\eta), \quad (2.109)$$

(5.134) and (5.155) yield that (C is a large enough positive number)

$$\lim_{M \rightarrow \infty} \|e^{2\xi/3} \Psi^a(\xi)(\xi - T)\|_{L^2([T, \infty))} < C, \quad (2.110)$$

$$\lim_{M \rightarrow \infty} \|e^{-2\eta/3} \tilde{\Psi}^a(\eta)(\eta - T)\|_{L^2([T, \infty))} = 0, \quad (2.111)$$

then by proposition 2.5, we have in trace norm

$$\lim_{M \rightarrow \infty} \chi(\xi) f_a(x) e^{2\xi/3} \tilde{K}_{2a}(\xi, \eta) f_a^{-1}(y) e^{-2\eta/3} \chi(\eta) = 0. \quad (2.112)$$

For the \tilde{K}_{2b} part, we have the formula similar to (2.99)

$$\begin{aligned} & f_a(x) e^{2\xi/3} \tilde{K}_{2b}(\xi, \eta) f_a^{-1}(y) e^{-2\eta/3} \\ &= \frac{1}{(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M}} \sum_{r'=1}^{r-r_s} e^{2\xi/3} \Psi_{r'}^a(\xi) \tilde{\Psi}_{r'}^a(\eta) e^{-2\eta/3} \\ &+ \sum_{j=r-r_s+1}^r \frac{1}{1+a} e^{2\xi/3} \Psi_{t', r'}^a(\xi) \tilde{\Psi}_{t', r'}^a(\eta) e^{-2\eta/3}, \end{aligned} \quad (2.113)$$

where

$$\Psi_{r'}^a(\xi) = (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}} \psi_{r'}(p+q\xi), \quad (2.114)$$

$$\tilde{\Psi}_{r'}^a(\eta) = (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a}y} \tilde{\psi}_{r'}(p+q\eta), \quad (2.115)$$

$$\Psi_{t',r'}^a(\xi) = \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{t'} \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}} \psi_{r'}(p+q\xi), \quad (2.116)$$

$$\tilde{\Psi}_{t',r'}^a(\eta) = \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{1-t'} (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a}y} \tilde{\psi}_{r'}(p+q\eta). \quad (2.117)$$

(5.133) and (5.155) yield that for $r' = 1, \dots, r - r_s$, (C is a large enough positive number)

$$\lim_{M \rightarrow \infty} \|e^{2\xi/3} \Psi_{r'}^a(\xi) \chi(\xi)\|_{L^2([T, \infty))} < C, \quad (2.118)$$

$$\lim_{M \rightarrow \infty} \|e^{-2\eta/3} \tilde{\Psi}_{r'}^a(\eta) \chi(\eta)\|_{L^2([T, \infty))} = 0, \quad (2.119)$$

so that similar to (2.102), we have the trace norm convergence

$$\lim_{M \rightarrow \infty} \left\| \frac{1}{(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M}} \chi(\xi) \left(\sum_{r'=1}^{r-r_s} e^{2\xi/3} \Psi_{r'}^a(\xi) e^{-2\eta/3} \tilde{\Psi}_{r'}^a(\eta) \right) \chi(\eta) \right\|_{\text{Tr}} = 0. \quad (2.120)$$

However, (5.132) and (5.154) give the L^2 norm convergence results

$$\lim_{M \rightarrow \infty} e^{2\xi/3} \Psi_{t',r'}^a(\xi) \chi(\xi) = \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a,r'-t'} (-1)^{t'-1} e^{2\xi/3} \frac{H_{t'-1}(x)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}, \quad (2.121)$$

$$\lim_{M \rightarrow \infty} e^{-2\eta/3} \tilde{\Psi}_{t',r'}^a(\eta) \chi(\eta) = \frac{(-a)^r}{(1+a)^{r'-1} \bar{C}_{r'-t'}} \frac{(-1)^{t'-1}}{(t'-1)!} e^{-2\eta/3} H_{t'-1}(\eta). \quad (2.122)$$

Then by proposition 2.4, we have the trace norm convergence result

$$\begin{aligned} \lim_{M \rightarrow \infty} \chi(\xi) \left(\sum_{j=r-r_s+1}^r \frac{1}{1+a} e^{2\xi/3} \Psi_{t',r'}^a(\xi) \tilde{\Psi}_{t',r'}^a(\eta) e^{-2\eta/3} \right) \chi(\eta) = \\ \chi(\xi) e^{2\xi/3} \left(\sum_{j=0}^{r_s-1} \frac{1}{j! \sqrt{2\pi}} H_j(\xi) H_j(\eta) e^{-\frac{\xi^2}{2}} \right) e^{-2\eta/3} \chi(\eta). \end{aligned} \quad (2.123)$$

Therefore,

$$\begin{aligned} & \lim_{M \rightarrow \infty} \det(1 - \chi(\xi) \tilde{K}(\xi, \eta) \chi(\eta)) \\ &= \lim_{M \rightarrow \infty} \det \left(1 - \chi(\xi) f_a(x) e^{2\xi/3} (\tilde{K}_{2a}(\xi, \eta) + \tilde{K}_{2b}(\xi, \eta)) f_a^{-1}(y) e^{-2\eta/3} \right) \\ &= \det \left(1 - \chi(\xi) e^{2\xi/3} \left(\sum_{j=0}^{r_s-1} \frac{1}{j! \sqrt{2\pi}} H_j(\xi) H_j(\eta) e^{-\frac{\xi^2}{2}} \right) e^{-2\eta/3} \chi(\eta) \right) \\ &= \det \left(1 - \chi(\xi) \left(\sum_{j=0}^{r_s-1} \frac{1}{j! \sqrt{2\pi}} H_j(\xi) e^{-\frac{\xi^2}{4}} H_j(\eta) e^{-\frac{\eta^2}{4}} \right) \chi(\eta) \right), \end{aligned} \quad (2.124)$$

where in the last step we conjugate the kernel by $e^{-\frac{2\xi}{3} + \frac{\xi^2}{4}}$.

Chapter 3

Rank 1 quaternionic spiked model

In this chapter, we consider the rank 1 quaternionic Wishart ensemble unless otherwise stated. Since we have only one parameter α_1 in the p.d.f. of sample eigenvalues (1.52), we denote it as a .

The reader should be cautious that some notations, for instance, φ_j and ψ_j , are defined differently from their definitions in chapter 2, and $\tilde{V}^4(\lambda)$ is not the fourth power of $\tilde{V}(\lambda)$.

3.1 The joint distribution function

In this section, we prove

Theorem 3.1. *The joint probability distribution function of λ in the quaternionic spiked model is*

$$P(\lambda) = \frac{1}{C} \tilde{V}^4(\lambda) \prod_{j=1}^N \left(\lambda_j^{2(M-N)+1} e^{-2M\lambda_j} \right). \quad (3.1)$$

Here

$$\tilde{V}^4(\lambda) = \begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & (2N-2)\lambda_1^{2N-3} & \dots & \lambda_N^{2N-2} & (2N-2)\lambda_N^{2N-3} \\ e^{\frac{a}{1+a}2M\lambda_1} & \frac{a}{1+a}2Me^{\frac{a}{1+a}2M\lambda_1} & \dots & e^{\frac{a}{1+a}2M\lambda_N} & \frac{a}{1+a}2Me^{\frac{a}{1+a}2M\lambda_N} \end{vmatrix}, \quad (3.2)$$

the determinant of a $2N \times 2N$ matrix whose $(2N, 2k-1)$ entry is $e^{\frac{a}{1+a}2M\lambda_k}$, $(j, 2k-1)$ entry is λ_k^{j-1} for $j = 1, \dots, 2N-1$, and $2i$ -th column is the derivative of the $(2i-1)$ -st column. $\tilde{V}^4(\lambda)$ is a variation of the $V(\lambda)^4$ appearing in the LSE (see [21]).

Since rank $r = 1$, we can simplify (1.52) as

$$P(\lambda) = \frac{1}{C} V(\lambda)^4 \prod_{j=1}^N \lambda_j^{2(M-N)+1} e^{-2M\lambda_j} \sum_{j=0}^{\infty} \frac{(2M)^j}{j!} \frac{Q_{(j)}(\frac{a}{1+a}) Q_{(j)}(\lambda_1, \dots, \lambda_N)}{Q_{(j)}(I_N)}, \quad (3.3)$$

just like formula (2.5) for the complex case.

We have [26]

$$Q_{(j)}(\frac{a}{1+a}) = \left(\frac{a}{1+a} \right)^j \quad (3.4)$$

and since the number of variables is N [26]

$$Q_{(j)}(1, \dots, 1) = \frac{1}{(j+1)!} \prod_{i=0}^{j-1} (2N+i), \quad (3.5)$$

so we get

$$\sum_{j=0}^{\infty} \frac{(2M)^j}{j!} \frac{Q_{(j)}(\frac{a}{1+a})Q_{(j)}(\lambda_1, \dots, \lambda_N)}{Q_{(j)}(I_N)} = \sum_{j=0}^{\infty} \frac{j+1}{\prod_{i=0}^{j-1}(2N+i)} \left(\frac{a}{1+a} 2M \right)^j Q_{(j)}(\lambda_1, \dots, \lambda_N). \quad (3.6)$$

In [26] there is an identity

$$\sum_{j=0}^{\infty} (j+1)Q_{(j)}(\lambda_1, \dots, \lambda_N)t^j = \prod_{j=1}^N \frac{1}{(1+\lambda_j t)^2}. \quad (3.7)$$

Comparing it with the well-known identity for Schur polynomials

$$\sum_{j=0}^{\infty} s_{(j)}(\lambda_1, \dots, \lambda_N)t^j = \prod_{j=1}^N \frac{1}{1+\lambda_j t}, \quad (3.8)$$

we get the identity

$$(j+1)Q_{(j)}(\lambda_1, \dots, \lambda_N) = s_{(j)}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_N, \lambda_N), \quad (3.9)$$

with each λ_i appearing twice as variables of the $s_{(j)}$. For notational simplicity, we denote the right hand side of (3.9) as $\tilde{s}_{(j)}(\Lambda)$, which is a plethysm [19]

$$\tilde{s}_{(j)}(\lambda_1, \dots, \lambda_N) = s_{(j)} \circ 2p_1(\lambda_1, \dots, \lambda_N). \quad (3.10)$$

Now we get

$$\sum_{j=0}^{\infty} \frac{(2M)^j}{j!} \frac{Q_{(j)}(\frac{a}{1+a})Q_{(j)}(\lambda_1, \dots, \lambda_N)}{Q_{(j)}(I_N)} = \sum_{j=0}^{\infty} \frac{1}{\prod_{i=0}^{j-1}(2N+i)} \left(\frac{a}{1+a} 2M \right)^j \tilde{s}_{(j)}(\lambda_1, \dots, \lambda_N). \quad (3.11)$$

Then we need a lemma to simplify (3.11) further.

Lemma 3.1.

$$\tilde{s}_{(j)}(\Lambda) = \frac{\begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & (2N-2)\lambda_1^{2N-3} & \dots & \lambda_N^{2N-2} & (2N-2)\lambda_N^{2N-3} \\ \lambda_1^{2N+j-1} & (2N+j-1)\lambda_1^{2N+j-2} & \dots & \lambda_N^{2N+j-1} & (2N+j-1)\lambda_N^{2N+j-2} \end{vmatrix}}{V(\lambda)^4}, \quad (3.12)$$

with the $(k, 2j-1)$ entry of the matrix being a power of λ_j with the exponent $k-1$ if $k \neq 2N$ and $2N+j-1$ if $k = 2N$, and the $(k, 2j)$ entry being the derivative of the $(k, 2j-1)$ entry with respect to λ_j .

To prove this lemma, we need the well known fact (see [21]), proven by L'Hôpital's rule

$$V(\lambda)^4 = \begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-1} & (2N-1)\lambda_1^{2N-2} & \dots & \lambda_N^{2N-1} & (2N-1)\lambda_N^{2N-2} \end{vmatrix}, \quad (3.13)$$

with the $(k, 2j-1)$ entry being λ_j^{k-1} and the $(k, 2j)$ entry $(k-1)\lambda_j^{k-2}$.

Proof of the lemma. Applying the L'Hôpital's rule repeatedly with respect to x_{2i} ,

$i = 1, \dots, N$, we get the identity

$$\begin{aligned}
 & \left| \begin{array}{ccccc} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & (2N-2)\lambda_1^{2N-3} & \dots & \lambda_N^{2N-2} & (2N-2)\lambda_N^{2N-3} \\ \lambda_1^{2N+j-1} & (2N+j-1)\lambda_1^{2N+j-2} & \dots & \lambda_N^{2N+j-1} & (2N+j-1)\lambda_N^{2N+j-2} \end{array} \right| \\
 & \left| \begin{array}{ccccc} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-1} & (2N-1)\lambda_1^{2N-2} & \dots & \lambda_N^{2N-1} & (2N-1)\lambda_N^{2N-2} \end{array} \right| \\
 & \left| \begin{array}{ccccc} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_{2N-1} & x_{2N} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ x_1^{2N-2} & x_2^{2N-2} & \dots & x_{2N-1}^{2N-2} & x_{2N}^{2N-2} \\ x_1^{2N+j-1} & x_2^{2N+j-1} & \dots & x_{2N-1}^{2N+j-1} & x_{2N}^{2N+j-1} \end{array} \right| \\
 = & \frac{\frac{\partial^N}{\partial x_2 \partial x_4 \dots \partial x_{2N}}}{\frac{\partial^N}{\partial x_2 \partial x_4 \dots \partial x_{2N}}} \left| \begin{array}{ccccc} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_{2N-1} & x_{2N} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ x_1^{2N-2} & x_2^{2N-2} & \dots & x_{2N-1}^{2N-2} & x_{2N}^{2N-2} \\ x_1^{2N+j-1} & x_2^{2N+j-1} & \dots & x_{2N-1}^{2N+j-1} & x_{2N}^{2N+j-1} \end{array} \right|_{\substack{x_{2i-1}=x_{2i}=\lambda_i \\ i=1, \dots, N}}
 \end{aligned} \tag{3.14}$$

$$= s_{(j)}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_N, \lambda_N) = \tilde{s}_{(j)}(\lambda_1, \dots, \lambda_N),$$

from the matrix representation of Schur polynomials, and now use (3.13) to get the compact formula (3.12). \square

Substituting (3.12) into (3.11), we get

$$\begin{aligned}
& V(\lambda)^4 \sum_{j=0}^{\infty} \frac{(2M)^j}{j!} \frac{Q_{(j)}\left(\frac{a}{1+a}\right) Q_{(j)}(\lambda_1, \dots, \lambda_N)}{Q_{(j)}(I_N)} \\
&= \begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & (2N-2)\lambda_1^{2N-3} & \dots & \lambda_N^{2N-2} & (2N-2)\lambda_N^{2N-3} \\ p(\lambda_1) & p'(\lambda_1) & \dots & p(\lambda_N) & p'(\lambda_N) \end{vmatrix} \\
&= \frac{1}{C} \begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & (2N-2)\lambda_1^{2N-3} & \dots & \lambda_N^{2N-2} & (2N-2)\lambda_N^{2N-3} \\ e^{\frac{a}{1+a}2M\lambda_1} & \frac{a}{1+a}2Me^{\frac{a}{1+a}2M\lambda_1} & \dots & e^{\frac{a}{1+a}2M\lambda_N} & \frac{a}{1+a}2Me^{\frac{a}{1+a}2M\lambda_N} \end{vmatrix} \\
&= \frac{1}{C} \tilde{V}^4(\lambda),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
p(x) &= \sum_{j=0}^{\infty} \frac{1}{\prod_{i=0}^{j-1} (2N+i)} \left(\frac{a}{1+a} 2M \right)^j x^{2N+j-1} \\
&= \frac{(2N-1)!}{\left(\frac{a}{1+a} 2M \right)^{2N-1}} \left(e^{\frac{a}{1+a} 2Mx} - \sum_{j=0}^{2N-2} \frac{1}{j!} \left(\frac{a}{1+a} 2Mx \right)^j \right),
\end{aligned} \tag{3.16}$$

and if $k \neq 2N$, the $(k, 2j-1)$ entries in both matrices are λ_j^{k-1} , and the $(k, 2j)$ entries are $(k-1)\lambda_j^{k-2}$, and the $2N, 2i-1$ entry in the former (latter) matrix is $p(\lambda_i)$ (resp. $e^{\frac{a}{1+a}2M\lambda_i}$) and the $2N, 2i$ entry $p'(\lambda_i)$ (resp. $\frac{a}{1+a}2Me^{\frac{a}{1+a}2M\lambda_i}$).

Proof of the theorem. Formulas (3.3) and (3.15) together give the result (3.1). \square

3.2 The determinantal formula

With the formula (3.1) ready to use, we get the limiting distribution formula for the largest sample eigenvalue, in the same spirit as the solution of the quaternionic white Wishart ensemble. Our process below is closely parallel to that in [30].

First, we find a skew orthogonal basis $\{\varphi_0(x), \varphi_1(x), \dots, \varphi_{2N-1}(x)\}$ of the linear space spanned by $\{1, x, x^2, \dots, x^{2N-2}, e^{\frac{a}{1+a}2Mx}\}$. We require that the $\varphi_j(x)$ is a linear combination of $\{1, x, x^2, \dots, x^j\}$ if $j < 2N - 1$, while $\varphi_{2N-1}(x)$ can be arbitrary, with the skew inner products among them

$$\begin{aligned} \langle \varphi_j(x), \varphi_k(x) \rangle_4 &= \int_0^\infty (\varphi_j(x)\varphi'_k(x) - \varphi'_j(x)\varphi_k(x))x^{2(M-N)+1}e^{-2Mx}dx \\ &= \begin{cases} r_{j/2} & \text{if } j \text{ is even and } k = j + 1, \\ -r_{k/2} & \text{if } k \text{ is even and } j = k + 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.17)$$

Remark 3.1. Due to the shortage of notations, we abuse the language so that we use φ and ψ in this chapter to mean functions different from those in chapter 2.

Then we can reformulate the distribution function of λ as

$$\begin{aligned} P(\lambda) &= \frac{1}{C} \begin{vmatrix} \varphi_0(\lambda_1) & \varphi'_0(\lambda_1) & \dots & \varphi_0(\lambda_N) & \varphi'_0(\lambda_N) \\ \varphi_1(\lambda_1) & \varphi'_1(\lambda_1) & \dots & \varphi_1(\lambda_N) & \varphi'_1(\lambda_N) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \varphi_{2N-1}(\lambda_1) & \varphi'_{2N-1}(\lambda_1) & \dots & \varphi_{2N-1}(\lambda_N) & \varphi'_{2N-1}(\lambda_N) \end{vmatrix} \\ &\quad \prod_{j=1}^N \left(\lambda_j^{2(M-N)+1} e^{-2M\lambda_j} \right) \\ &= \frac{1}{C} \begin{vmatrix} \psi_0(\lambda_1) & \psi'_0(\lambda_1) & \dots & \psi_0(\lambda_N) & \psi'_0(\lambda_N) \\ \psi_1(\lambda_1) & \psi'_1(\lambda_1) & \dots & \psi_1(\lambda_N) & \psi'_1(\lambda_N) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \psi_{2N-1}(\lambda_1) & \psi'_{2N-1}(\lambda_1) & \dots & \psi_{2N-1}(\lambda_N) & \psi'_{2N-1}(\lambda_N) \end{vmatrix}, \end{aligned} \quad (3.18)$$

where

$$\psi_i(x) = \varphi_i(x)x^{M-N+1/2}e^{-Mx}. \quad (3.19)$$

For an arbitrary function $f(x)$ on $[0, \infty)$, by the formula of de Bruijn [8],

$$\int_0^\infty \cdots \int_0^\infty \begin{vmatrix} \psi_0(\lambda_1) & \psi'_0(\lambda_1) & \cdots & \psi_0(\lambda_N) & \psi'_0(\lambda_N) \\ \psi_1(\lambda_1) & \psi'_1(\lambda_1) & \cdots & \psi_1(\lambda_N) & \psi'_1(\lambda_N) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \psi_{2N-1}(\lambda_1) & \psi'_{2N-1}(\lambda_1) & \cdots & \psi_{2N-1}(\lambda_N) & \psi'_{2N-1}(\lambda_N) \end{vmatrix} \prod_{i=1}^N (1 + f(\lambda_i)) d\lambda_i = C \text{Pf}(P(1 + f)), \quad (3.20)$$

where $P(1 + f)$ is a $2N \times 2N$ matrix, whose entries depend on $1 + f$ in the following way

$$(P(1 + f))_{j,k} = \int_0^\infty (\psi_{j-1}(x)\psi'_{k-1}(x) - \psi'_{j-1}(x)\psi_{k-1}(x))(1 + f(x))dx. \quad (3.21)$$

Now we define a matrix Z as

$$Z = \begin{pmatrix} 0 & r_0 & & & & \\ -r_0 & 0 & & & & \\ & & 0 & r_1 & & \\ & & -r_1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & r_{N-1} \\ & & & & & -r_{N-1} & 0 \end{pmatrix}, \quad (3.22)$$

with

$$Z_{j,k} = \begin{cases} r_{k/2-1} & \text{if } k \text{ is even and } j = k - 1, \\ -r_{j/2-1} & \text{if } j \text{ is even and } k = j - 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.23)$$

and define for $j = 0, \dots, N-1$, $\eta = Z^{-1}\psi$, i.e.,

$$\eta_{2j}(x) = -\frac{\psi_{2j+1}(x)}{r_j} \quad \text{and} \quad \eta_{2j+1}(x) = \frac{\psi_{2j}(x)}{r_j}. \quad (3.24)$$

So we have

$$\begin{aligned} (P(1+f))_{j,k} &= \int_0^\infty (\psi_{j-1}(x)\psi'_{k-1}(x) - \psi'_{j-1}(x)\psi_{k-1}(x))dx \\ &\quad + \int_0^\infty (\psi_{j-1}(x)\psi'_{k-1}(x) - \psi'_{j-1}(x)\psi_{k-1}(x))f(x)dx \\ &= Z_{j,k} + \int_0^\infty (\psi_{j-1}(x)\psi'_{k-1}(x) - \psi'_{j-1}(x)\psi_{k-1}(x))f(x)dx. \end{aligned} \quad (3.25)$$

And if we denote $Q(1+f) = Z^{-1}P(1+f)$, then

$$Q(1+f)_{j,k} = \delta_{j,k} + \int_0^\infty (\eta_{j-1}(x)\psi'_{k-1}(x) - \eta'_{j-1}(x)\psi_{k-1}(x))f(x)dx. \quad (3.26)$$

If we choose f to be $-\chi_{(T,\infty)}$, then the integral on the left hand side of (3.20), after multiplying a constant, is the probability of all λ_i 's smaller than T . In latter part of the paper, we abbreviate $\chi_{(T,\infty)}$ to χ as before. So we get for a T -independent constant

$$\mathbb{P}(\max(\lambda_i) \leq T) = C \text{Pf}(P(1-\chi)), \quad (3.27)$$

and

$$(\mathbb{P}(\max(\lambda_i) \leq T))^2 = C^2 \det(P(1-\chi)) = C^2 \det(Q(1-\chi)). \quad (3.28)$$

Now we apply a matrix version of (2.51). In linear algebra, we have the determinant identity

$$\det(I - AB) = \det(I - BA), \quad (3.29)$$

for A an linear map from \mathbb{R}^n to \mathbb{R}^m and B an linear map from \mathbb{R}^m to \mathbb{R}^n , and the identity still holds in infinite dimensional settings [12]. Letting \det mean a Fredholm

determinant for a matrix integral operator defined in (1.71), we describe a setting due to Tracy-Widom [30].

If A is an operator from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to the vector space \mathbb{R}^{2N} with

$$A \begin{pmatrix} g(x) \\ h(x) \end{pmatrix}_j = \int_0^\infty \chi(x) \eta_{j-1}(x) g(x) dx - \int_0^\infty \chi(x) \eta'_{j-1}(x) h(x) dx, \quad (3.30)$$

and B is an operator from \mathbb{R}^{2N} to $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$B \begin{pmatrix} c_1 \\ \vdots \\ c_{2N} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{2N} c_k \psi'_{k-1}(x) \chi(x) \\ \sum_{k=1}^{2N} c_k \psi_{k-1}(x) \chi(x) \end{pmatrix}, \quad (3.31)$$

then

$$I - AB = Q(1 - \chi), \quad (3.32)$$

and

$$I - BA = I - \chi(x) \begin{pmatrix} S_4(x, y) & SD_4(x, y) \\ IS_4(x, y) & S_4(y, x) \end{pmatrix} \chi(y), \quad (3.33)$$

where $S_4(x, y)$, $IS_4(x, y)$ and $SD_4(x, y)$ are integral operators whose kernels are

$$S_4(x, y) = \sum_{j=0}^{2N-1} \psi'_j(x) \eta_j(y) = \sum_{j=0}^{N-1} \frac{1}{r_j} (-\psi'_{2j}(x) \psi_{2j+1}(y) + \psi'_{2j+1}(x) \psi_{2j}(y)), \quad (3.34)$$

$$SD_4(x, y) = \sum_{j=0}^{2N-1} -\psi'_j(x) \eta'_j(y) = \sum_{j=0}^{N-1} \frac{1}{r_j} (\psi'_{2j}(x) \psi'_{2j+1}(y) - \psi'_{2j+1}(x) \psi'_{2j}(y)), \quad (3.35)$$

$$IS_4(x, y) = \sum_{j=0}^{2N-1} \psi_j(x) \eta_j(y) = \sum_{j=0}^{N-1} \frac{1}{r_j} (-\psi_{2j}(x) \psi_{2j+1}(y) + \psi_{2j+1}(x) \psi_{2j}(y)), \quad (3.36)$$

$$S_4(y, x) = \sum_{j=0}^{2N-1} -\psi_j(x) \eta'_j(y) = \sum_{j=0}^{N-1} \frac{1}{r_j} (\psi_{2j}(x) \psi'_{2j+1}(y) - \psi_{2j+1}(x) \psi'_{2j}(y)). \quad (3.37)$$

Remark 3.2. It is clear that the nomenclature of $SD_4(x, y)$ is due to the fact that $SD_4(x, y)$ is the negative of the derivative of $S_4(x, y)$. But $IS_4(x, y)$, which gets its

name in the same way in earlier literature in GSE (e.g., [29]), in our problem may not satisfy the equation

$$IS_4(x, y) = - \int_x^\infty S_4(t, y) dt, \quad (3.38)$$

since the integral on the right hand side may diverge.

In conclusion,

$$(\mathbb{P}(\max(\lambda_i) \leq T))^2 = C^2 \det \left(I - \chi(x) \begin{pmatrix} S_4(x, y) & SD_4(x, y) \\ IS_4(x, y) & S_4(y, x) \end{pmatrix} \chi(y) \right), \quad (3.39)$$

and we can find that $C^2 = 1$ by taking the limit $T \rightarrow \infty$. We define a 2×2 matrix kernel as

$$\begin{aligned} P_T(x, y) &= \chi(x) \begin{pmatrix} S_4(x, y) & SD_4(x, y) \\ IS_4(x, y) & S_4(y, x) \end{pmatrix} \chi(y) \\ &= \begin{pmatrix} \chi(x)S_4(x, y)\chi(y) & \chi(x)DS_4(x, y)\chi(y) \\ \chi(x)IS_4(x, y)\chi(y) & \chi(x)S_4(y, x)\chi(y) \end{pmatrix}, \end{aligned} \quad (3.40)$$

then we have

$$(\mathbb{P}(\max(\lambda_i) \leq T))^2 = \det(I - P_T(x, y)). \quad (3.41)$$

3.3 $S_4(x, y)$ in terms of Laguerre polynomials

In manipulation of skew orthogonal polynomials, we take the approach of [2], and all properties of Laguerre polynomials are from [27].

Since Laguerre polynomials by definition satisfy the orthogonal property

$$\int_0^\infty L_j^{(2(M-N))} L_k^{(2(M-N))} x^{2(M-N)} e^{-x} dx = \frac{(j + 2(M - N))!}{j!} \delta_{j,k}, \quad (3.42)$$

and they have the differential identity ¹

$$x \frac{d}{dx} L_n^{(2(M-N))}(x) = n L_n^{(2(M-N))}(x) - (n + 2(M - N)) L_{n-1}^{(2(M-N))}(x), \quad (3.43)$$

it is easy to get that

$$\begin{aligned} & \left\langle L_j^{(2(M-N))}(2Mx), L_k^{(2(M-N))}(2Mx) \right\rangle_4 \\ &= \int_0^\infty \left(L_j^{(2(M-N))}(2Mx) \frac{d}{dx} L_k^{(2(M-N))}(2Mx) \right. \\ & \quad \left. - L_k^{(2(M-N))}(2Mx) \frac{d}{dx} L_j^{(2(M-N))}(2Mx) \right) x^{2(M-N)+1} e^{-2Mx} dx \\ &= \begin{cases} \left(\frac{1}{2M} \right)^{2(M-N)+1} \frac{(j+2(M-N))!}{(j-1)!} & \text{if } j = k + 1, \\ - \left(\frac{1}{2M} \right)^{2(M-N)+1} \frac{(k+2(M-N))!}{(k-1)!} & \text{if } k = j + 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.44)$$

So we can choose for $j = 0, \dots, N - 2$,

$$\varphi_{2j}(x) = \sum_{k=0}^j \left(\prod_{i=1}^k \frac{2i-1}{2i+2(M-N)} \right) L_{2k}^{(2(M-N))}(2Mx), \quad (3.45)$$

$$\varphi_{2j+1}(x) = - L_{2j+1}^{(2(M-N))}(2Mx), \quad (3.46)$$

and

$$r_j = \left(\frac{1}{2M} \right)^{2(M-N)+1} \frac{(2j+2(M-N)+1)!}{(2j)!} \prod_{k=1}^j \frac{2k-1}{2k+2(M-N)}. \quad (3.47)$$

We can also choose

$$\varphi_{2N-2}(x) = \sum_{k=0}^{N-1} \left(\prod_{i=1}^k \frac{2i-1}{2i+2(M-N)} \right) L_{2k}^{(2(M-N))}(2Mx), \quad (3.48)$$

¹We assume $L_n^{(2(M-N))}(x) = 0$ if $n < 0$.

but $\varphi_{2N-1}(x)$ is not a polynomial and needs to be treated separately.

By the Rodrigues' representation

$$x^{2(M-N)} e^{-x} L_n^{(2(M-N))}(x) = \frac{1}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+2(M-N)}), \quad (3.49)$$

and repeated integration by parts, we get for $n > 0$

$$\begin{aligned} \left\langle e^{\frac{a}{1+a} 2Mx}, L_n^{(2(M-N))}(2Mx) \right\rangle_4 = \\ \left(\frac{1+a}{2M} \right)^{2(M-N)+1} \left((-a)^{n+1} \frac{(n+2(M-N)+1)!}{n!} - (-a)^{n-1} \frac{(n+2(M-N))!}{(n-1)!} \right) \end{aligned} \quad (3.50)$$

and

$$\left\langle e^{\frac{a}{1+a} 2Mx}, L_0^{(2(M-N))}(2Mx) \right\rangle_4 = - \left(\frac{1+a}{2M} \right)^{2(M-N)+1} a(2(M-N)+1)!, \quad (3.51)$$

so that

$$\begin{aligned} \left\langle e^{\frac{a}{1+a} 2Mx}, \varphi_{2j}(x) \right\rangle_4 = \\ - \left(\frac{1+a}{2M} \right)^{2(M-N)+1} a^{2j+1} \frac{(2j+2(M-N)+1)!}{(2j)!} \prod_{k=1}^j \frac{2k-1}{2k+2(M-N)} \end{aligned} \quad (3.52)$$

and

$$\begin{aligned} \left\langle e^{\frac{a}{1+a} 2Mx}, \varphi_{2j+1}(x) \right\rangle_4 = \\ - \left(\frac{1+a}{2M} \right)^{2(M-N)+1} \left(a^{2j+2} \frac{(2j+2(M-N)+2)!}{(2j+1)!} - a^{2j} \frac{(2j+2(M-N)+1)!}{(2j)!} \right). \end{aligned} \quad (3.53)$$

Now by the skew orthogonality, we can choose

$$\begin{aligned}
\varphi_{2N-1}(x) &= - \sum_{j=0}^{N-2} \frac{1}{r_j} \left(\left\langle e^{\frac{a}{1+a}2Mx}, \varphi_{2j+1}(x) \right\rangle_4 \varphi_{2j}(x) - \left\langle e^{\frac{a}{1+a}2Mx}, \varphi_{2j}(x) \right\rangle_4 \varphi_{2j+1}(x) \right) \\
&\quad - (1+a)^{2(M-N)+1} a^{2N-2} \prod_{j=1}^{N-1} \frac{2j+2(M-N)}{2j-1} \varphi_{2N-2}(x) + e^{\frac{a}{1+a}2Mx} \\
&= e^{\frac{a}{1+a}2Mx} - (1+a)^{2(M-N)+1} \sum_{j=0}^{2N-2} (-a)^j L_j^{(2(M-N))}(2Mx)
\end{aligned} \tag{3.54}$$

and

$$r_{N-1} = \left(\frac{1+a}{2M} \right)^{2(M-N)+1} a^{2N-1} \frac{(2M-1)!}{(2N-2)!} \prod_{k=1}^{N-1} \frac{2k-1}{2k+2(M-N)}. \tag{3.55}$$

Now, we write $S_4(x, y)$ as $S_{4a}(x, y) + S_{4b}(x, y)$, where

$$S_{4a}(x, y) = \sum_{j=0}^{N-2} \frac{1}{r_j} (-\psi'_{2j}(x) \psi_{2j+1}(y) + \psi'_{2j+1}(x) \psi_{2j}(y)) \tag{3.56}$$

and

$$S_{4b}(x, y) = \frac{1}{r_{N-1}} (-\psi'_{2N-2}(x) \psi_{2N-1}(y) + \psi'_{2N-1}(x) \psi_{2N-2}(y)), \tag{3.57}$$

and simplify them separately.

The formula (3.56) of our $S_{4a}(x, y)$ is also the formula for $S_4(x, y)$ in the LSE problem, with parameters M and $N-2$, and has been well studied. For completeness we derive its Laguerre polynomial expression here, following [2].

By the differential identity (3.43) and the identity

$$nL_n^{(2(M-N))}(x) = (-x + 2n + 2(M - N) - 1)L_{n-1}^{(2(M-N))}(x) - (n + 2(M - N) - 1)L_{n-2}^{(2(M-N))}(x), \quad (3.58)$$

we get, remembering the definition (3.19), the telescoping sequence

$$\begin{aligned} \psi'_{2j}(x) &= \sum_{k=0}^j \left(\prod_{i=1}^k \frac{2i-1}{2i+2(M-N)} \right. \\ &\quad \left. \left(M - N + 1/2 - Mx + x \frac{d}{dx} \right) L_{2k}^{(2(M-N))}(2Mx) \right) x^{M-N-1/2} e^{-Mx} \\ &= \frac{1}{2} \sum_{k=0}^j \prod_{i=1}^k \frac{2i-1}{2i+2(M-N)} \left((2k+1)L_{2k+1}^{(2(M-N))}(2Mx) \right. \\ &\quad \left. - (2k+2(M-N))L_{2k-1}^{(2(M-N))}(2Mx) \right) x^{M-N-1/2} e^{-Mx} \\ &= \frac{1}{2} \left(\prod_{k=1}^j \frac{2k-1}{2k+2(M-N)} \right) (2j+1)L_{2j+1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} \end{aligned} \quad (3.59)$$

and

$$\begin{aligned} \psi'_{2j+1}(x) &= - \left(M - N + 1/2 - Mx + x \frac{d}{dx} \right) L_{2j+1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} \\ &= - \frac{1}{2} \left((2j+2)L_{2j+2}^{(2(M-N))}(2Mx) \right. \\ &\quad \left. - (2j+2(M-N)+1)L_{2j}^{(2(M-N))}(2Mx) \right) x^{M-N-1/2} e^{-Mx}. \end{aligned} \quad (3.60)$$

Therefore, if we plug in (3.47), (3.59) and (3.60) into (3.56), we get after some trick,

$$S_{4a}(x, y) = \frac{1}{2} (2M)^{2(M-N)+1} x^{M-N-1/2} e^{-Mx} y^{M-N+1/2} e^{-My} \left\{ \sum_{j=0}^{2N-2} \frac{j!}{(j+2(M-N))!} L_j^{(2(M-N))}(2Mx) L_j^{(2(M-N))}(2My) - \frac{(2N-2)!}{(2M-2)!} \left(\prod_{j=1}^{N-1} \frac{2j+2(M-N)}{2j-1} \right) L_{2N-2}^{(2(M-N))}(2Mx) \varphi_{2N-2}(y) \right\}. \quad (3.61)$$

Furthermore, we can simplify $\psi_{2N-2}(x)$. Since for $j \neq 2N-1$, (if we define $\varphi_j(x)$ and then $\psi_j(x)$ for $j > 2N-1$ by the formula (3.45) and (3.46),)

$$\int_0^\infty (\psi_{2N-2}(x) \psi'_j(x) - \psi'_{2N-2}(x) \psi_j(x)) dx = 0, \quad (3.62)$$

we get for $j \neq 2N-1$, using integration by parts,

$$\int_0^\infty \psi'_{2N-2}(x) L_j^{(2(M-N))}(2Mx) x^{M-N+1/2} e^{-Mx} dx = 0. \quad (3.63)$$

So by the orthogonal property of Laguerre polynomials, we get

$$\psi'_{2N-2}(x) = C L_{2N-1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx}, \quad (3.64)$$

and we can determine that

$$C = \frac{2N-1}{2} \prod_{j=1}^{N-1} \frac{2j-1}{2j+2(M-N)} \quad (3.65)$$

without much difficulty. Together with the fact $\lim_{x \rightarrow \infty} \psi_{2N-2}(x) = 0$, we get

$$\psi_{2N-2}(x) = -\frac{2N-1}{2} \prod_{j=1}^{N-1} \frac{2j-1}{2j+2(M-N)} \int_x^\infty t^{M-N-1/2} e^{-Mt} L_{2N-1}^{(2(M-N))}(2Mt) dt. \quad (3.66)$$

Now, we can write $S_{4a}(x, y)$ as $S_{4a1}(x, y) + S_{4a2}(x, y)$, where

$$S_{4a1}(x, y) = \frac{1}{2} (2M)^{2(M-N)+1} x^{M-N-1/2} e^{-Mx} y^{M-N+1/2} e^{-My} \sum_{j=0}^{2N-2} \frac{j!}{(j+2(M-N))!} L_j^{(2(M-N))}(2Mx) L_j^{(2(M-N))}(2My) \quad (3.67)$$

and

$$S_{4a2}(x, y) = \frac{1}{4} (2M)^{2(M-N)+1} \frac{(2N-1)!}{(2M-2)!} L_{2N-2}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} \int_y^\infty t^{M-N-1/2} e^{-Mt} L_{2N-1}^{(2(M-N))}(2Mt) dt. \quad (3.68)$$

Finally,

$$S_{4b}(x, y) = -\frac{1}{2} \left(\frac{2M}{1+a} \right)^{2(M-N)+1} a^{-(2N-1)} \frac{(2N-1)!}{(2M-1)!} \left\{ L_{2N-1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} \psi_{2N-1}(y) + \psi'_{2N-1}(x) \int_y^\infty L_{2N-1}^{(2(M-N))}(2Mt) t^{M-N-1/2} e^{-Mt} dt \right\}, \quad (3.69)$$

and we can take the asymptotic analyses of $S_{4a1}(x, y)$, $S_{4a2}(x, y)$ and $S_{4b}(x, y)$ separately.

3.4 Proof of theorem 1.2

The same as in the complex case, we consider the rescaled distribution problem, and wish to find the probability of the largest sample eigenvalue being in the domain $(0, p + qT]$. We can put the kernel in the new coordinate system (after a conjugation by $\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$), and get

$$\begin{aligned} (\mathbb{P}(\max(\lambda_i) \leq p + qT))^2 &= \det \left(I - \begin{pmatrix} \widetilde{S}_4(\xi, \eta) & \widetilde{SD}_4(\xi, \eta) \\ \widetilde{IS}_4(\xi, \eta) & \widetilde{S}_4(\eta, \xi) \end{pmatrix} \chi(\eta) \right) \\ &= \det(I - \widetilde{P}_T(\xi, \eta)), \end{aligned} \quad (3.70)$$

where as L^2 functions,

$$\widetilde{SD}_4(\xi, \eta) = q^2 SD_4(x, y)|_{\substack{x=p+q\xi, \\ y=p+q\eta}}, \quad (3.71)$$

$$\widetilde{S}_4(\xi, \eta) = q S_4(x, y)|_{\substack{x=p+q\xi, \\ y=p+q\eta}}, \quad (3.72)$$

$$\widetilde{IS}_4(\xi, \eta) = I S_4(x, y)|_{\substack{x=p+q\xi, \\ y=p+q\eta}}, \quad (3.73)$$

and

$$\widetilde{P}_T(\xi, \eta) = \chi(\xi) \begin{pmatrix} \widetilde{S}_4(\xi, \eta) & \widetilde{SD}_4(\xi, \eta) \\ \widetilde{IS}_4(\xi, \eta) & \widetilde{S}_4(\eta, \xi) \end{pmatrix} \chi(\eta). \quad (3.74)$$

In the proof of theorem 1.2, we need the matrix version of propositions 2.3—2.6, and the fact that the convergence in trace norm of a matrix integral operator is equivalent to the convergence in trace norm of all its entries.

Since the $IS_4(x, y)$ and $DS_4(x, y)$ are of the same form as $S_4(x, y)$, we only show the asymptotic analysis of $S_4(x, y)$, and state the result for the other two, for which the arguments are the same. We give proofs of all the three parts below.

3.4.1 The $-1 < a < \gamma^{-1}$ part

In case $-1 < a \leq \gamma^{-1}$, we choose $p = (1 + \gamma^{-1})^2$ and $q = \frac{(1+\gamma)^{4/3}}{\gamma(2M)^{2/3}}$, and denote ²

$$\tilde{S}_*(\xi, \eta) = \frac{(1 + \gamma)^{4/3}}{\gamma(2M)^{2/3}} S_*(x, y) \Big|_{\substack{x=(1+\gamma^{-1})^2 + \frac{(1+\gamma)^{4/3}}{\gamma(2M)^{2/3}} \xi \\ y=(1+\gamma^{-1})^2 + \frac{(1+\gamma)^{4/3}}{\gamma(2M)^{2/3}} \eta}}. \quad (3.75)$$

$S_{4a}(x, y)$ is the formula for the upper-left entry of the 2×2 matrix kernel of the quaternionic white Wishart ensemble with parameters M and $N - 1$, and its asymptotic behavior is well studied [11]. We want to prove that as $M \rightarrow \infty$, $S_{4a}(x, y)$ dominates $S_4(x, y)$ in the domain that we are interested in, and so naturally the distribution of the largest sample eigenvalue in the perturbed problem is the same as that in the quaternionic white Wishart ensemble. (The difference between N and $N - 1$ is negligible.)

$S_{4a1}(x, y)$ is almost the kernel for the complex white Wishart ensemble with parameters $2M - 2$ and $2N - 2$, besides a factor $\sqrt{y/x}/2$. By arguments in subsection 2.3.1 we have

$$\lim_{M \rightarrow \infty} \chi(\xi) \tilde{S}_{4a1}(\xi, \eta) \chi(\eta) = \frac{1}{2} \chi(\xi) K_{\text{Airy}}(\xi, \eta) \chi(\eta). \quad (3.76)$$

For the $S_{4a2}(x, y)$ part, we also have in trace norm [11],

$$\lim_{M \rightarrow \infty} \chi(\xi) \tilde{S}_{4a2}(\xi, \eta) \chi(\eta) = -\frac{1}{4} \chi(\xi) \text{Ai}(\xi) \int_{\eta}^{\infty} \text{Ai}(t) dt \chi(\eta). \quad (3.77)$$

We get the proof of (3.77) by asymptotics analysis. Formula (5.165) and a similar result for $L_{2N-2}^{(2(M-N))}$ imply the convergence in L^2 norm of functions in ξ and respectively

²Here $*$ stands for 4, $4a$, $4a1$, $4a2$ and $4b$. The definition of $\tilde{S}_*(\xi, \eta)$ in (3.75) is only used in subsection 3.4.1 and 3.4.2.

η ,

$$\lim_{M \rightarrow \infty} \gamma^{-2N} (1 + \gamma)^{4/3} (2M)^{1/3} e^{M-N} L_{2N-2}^{(2(M-N))} (2Mx) x^{M-N-1/2} e^{-Mx} \chi(\xi) = \text{Ai}(\xi) \chi(\xi), \quad (3.78)$$

$$\lim_{M \rightarrow \infty} \gamma^{-2N} 2M e^{M-N} \int_y^\infty L_{2N-1}^{(2(M-N))} (2Mt) t^{M-N-1/2} e^{-Mt} dt \chi(\eta) = - \int_\eta^\infty \text{Ai}(t) dt \chi(\eta), \quad (3.79)$$

and by the Stirling's formula,

$$\lim_{M \rightarrow \infty} (2M)^{2(M-N)-1} \frac{(2N-1)!}{(2M-2)!} e^{2(N-M)} \gamma^{4N-1} = 1. \quad (3.80)$$

By (3.68) and (3.75), we get

$$\begin{aligned} \chi(\xi) \tilde{S}_{4a2}(\xi, \eta) \chi(\eta) &= \frac{1}{4} (2M)^{2(M-N)-1} \frac{(2N-1)!}{(2M-2)!} e^{2(N-M)} \gamma^{4N-1} \\ &\quad \gamma^{-2N} (1 + \gamma)^{4/3} (2M)^{1/3} e^{M-N} L_{2N-2}^{(2(M-N))} (2Mx) x^{M-N-1/2} e^{-Mx} \chi(\xi) \\ &\quad \gamma^{-2N} 2M e^{M-N} \int_y^\infty L_{2N-1}^{(2(M-N))} (2Mt) t^{M-N-1/2} e^{-Mt} dt \chi(\eta). \end{aligned} \quad (3.81)$$

Therefore we get the trace norm convergence (3.77) from the L^2 convergence (3.78) and (3.79) by proposition 2.4.

Now we need to analyze the term $S_{4b}(\xi, \eta)$, new to the perturbed problem. We need the following results of L^2 convergence, which are direct consequences of (5.165), (5.172) and a similar result of ψ'_{2N-1} :

$$\begin{aligned} \lim_{M \rightarrow \infty} \gamma^{-2N-1} (1 + \gamma)^{4/3} (2M)^{1/3} e^{M-N} L_{2N-1}^{(2(M-N))} (2Mx) x^{M-N-1/2} e^{-Mx} \chi(\xi) = \\ - \text{Ai}(\xi) \chi(\xi), \end{aligned} \quad (3.82)$$

$$\lim_{M \rightarrow \infty} \gamma^{-2N} 2M e^{M-N} \int_y^\infty L_{2N-1}^{(2(M-N))}(2Mt) t^{M-N-1/2} e^{-Mt} dt \chi(\eta) = - \int_\eta^\infty \text{Ai}(t) dt \chi(\eta), \quad (3.83)$$

$$\lim_{M \rightarrow \infty} (1+a)^{2(N-M)-1} a^{-2N+1} \frac{(1-a\gamma)(2M)^{1/3}}{(\gamma+1)^{2/3} \gamma^{2N-1}} e^{M-N} \psi_{2N-1}(y) \chi(\eta) = \text{Ai}(\eta) \chi(\eta), \quad (3.84)$$

$$\lim_{M \rightarrow \infty} (1+a)^{2(N-M)-1} a^{-2N+1} \frac{(1-a\gamma)(\gamma+1)^{2/3}}{\gamma^{2N} (2M)^{1/3}} e^{M-N} \psi'_{2N-1}(y) \chi(\xi) = \text{Ai}'(\xi) \chi(\xi). \quad (3.85)$$

By the Stirling's formula, we get

$$\lim_{M \rightarrow \infty} (2M)^{2(M-N)} \frac{(2N-1)!}{(2M-1)!} e^{2(N-M)} \gamma^{4N-1} = 1, \quad (3.86)$$

and then by (3.69), (3.75) and proposition 2.4, we have the convergence in trace norm

$$\begin{aligned} \lim_{M \rightarrow \infty} \frac{(1-a\gamma)(2M)^{1/3}}{(1+\gamma)^{2/3}} \chi(\xi) \tilde{S}_{4b}(\xi, \eta) \chi(\eta) = \\ \frac{1}{2} \chi(\xi) \left(\text{Ai}(\xi) \text{Ai}(\eta) + \text{Ai}'(\xi) \int_\eta^\infty \text{Ai}(t) dt \right) \chi(\eta), \end{aligned} \quad (3.87)$$

which implies that in trace norm,

$$\lim_{M \rightarrow \infty} \chi(\xi) \tilde{S}_{4b}(\xi, \eta) \chi(\eta) = 0. \quad (3.88)$$

Now we get the desired result

$$\lim_{M \rightarrow \infty} \chi(\xi) \tilde{S}_4(\xi, \eta) \chi(\eta) = \lim_{M \rightarrow \infty} \chi(\xi) \tilde{S}_{4a}(\xi, \eta) \chi(\eta) = \chi(\xi) \hat{S}_4(\xi, \eta) \chi(\eta), \quad (3.89)$$

and in the same way

$$\lim_{M \rightarrow \infty} \chi(\xi) \widetilde{SD}_4(\xi, \eta) \chi(\eta) = \chi(\xi) \widehat{SD}_4(\xi, \eta) \chi(\eta), \quad (3.90)$$

$$\lim_{M \rightarrow \infty} \chi(\xi) \widetilde{IS}_4(\xi, \eta) \chi(\eta) = \chi(\xi) \widehat{IS}_4(\xi, \eta) \chi(\eta). \quad (3.91)$$

Therefore, in trace norm

$$\lim_{M \rightarrow \infty} \widetilde{P}_T(\xi, \eta) = \chi(\xi) \begin{pmatrix} \widehat{S}_4(\xi, \eta) & \widehat{SD}_4(\xi, \eta) \\ \widehat{IS}_4(\xi, \eta) & \widehat{S}_4(\eta, \xi) \end{pmatrix} \chi(\eta), \quad (3.92)$$

and the convergence of Fredholm determinant follows.

3.4.2 The $a = \gamma^{-1}$ part

When $a = \gamma^{-1}$, the $1 - a\gamma^{-1}$ in (3.87) vanishes, so we need other asymptotic formulas for $\psi_{2N-1}(\eta)$ and $\psi'_{2N-1}(\eta)$. The approach is similar to that in the $a < \gamma^{-1}$ case, with the same choice of p and q . We need the L^2 convergence results given by (5.172) and similar results:

$$\begin{aligned} \lim_{M \rightarrow \infty} \gamma^{-2N-1} (1 + \gamma)^{4/3} (2M)^{1/3} e^{M-N} e^{\xi/3} L_{2N-1}^{(2(M-N))} (2Mx) x^{M-N-1/2} e^{-Mx} \chi(\xi) = \\ - e^{\xi/3} \text{Ai}(\xi) \chi(\xi), \end{aligned} \quad (3.93)$$

$$\begin{aligned} \lim_{M \rightarrow \infty} \gamma^{-2N} 2M e^{M-N} e^{-\eta/3} \int_y^\infty L_{2N-1}^{(2(M-N))} (2Mt) t^{M-N-1/2} e^{-Mt} dt \chi(\eta) = \\ - e^{-\eta/3} \int_\eta^\infty \text{Ai}(t) dt \chi(\eta), \end{aligned} \quad (3.94)$$

$$\lim_{M \rightarrow \infty} (1+a)^{2(N-M)-1} a^{-2N+1} e^{M-N} \gamma^{-2N+1} e^{-\eta/3} \psi_{2N-1}(y) \chi(\eta) = e^{-\eta/3} s^{(1)}(\eta) \chi(\eta), \quad (3.95)$$

$$\lim_{M \rightarrow \infty} (1+a)^{2(N-M)-1} a^{-2N+1} e^{M-N} \frac{(\gamma+1)^{4/3}}{\gamma^{2N}(2M)^{-2/3}} e^{\xi/3} \psi'_{2N-1}(x) \chi(\xi) = e^{\xi/3} \text{Ai}(\xi) \chi(\xi). \quad (3.96)$$

Now we conclude the proof of the $a = \gamma^{-1}$ part of theorem 1.2. Using (3.69), (3.86) and proposition 2.4 we have the convergence in trace norm

$$\begin{aligned} \lim_{M \rightarrow \infty} \chi(\xi) e^{\xi/3} \widetilde{S}_{4b}(\xi, \eta) e^{-\eta/3} \chi(\eta) &= \frac{1}{2} \chi(\xi) e^{\xi/3} \left(\text{Ai}(\xi) s^{(1)}(\eta) + \text{Ai}(\xi) \int_{\eta}^{\infty} \text{Ai}(t) dt \right) e^{-\eta/3} \chi(\eta) \\ &= \frac{1}{2} \chi(\xi) e^{\xi/3} \text{Ai}(\xi) e^{-\eta/3} \chi(\eta), \end{aligned} \quad (3.97)$$

and this together with the conjugated convergence result of $\widetilde{S}_{4a}(\xi, \eta)$ in formulas (3.76) and (3.77) of subsection 3.4.1, which can be proved by arguments in subsection 2.3.2, conclude

$$\lim_{M \rightarrow \infty} \chi(\xi) e^{\xi/3} \widetilde{S}_4(\xi, \eta) e^{-\eta/3} \chi(\eta) = \chi(\xi) e^{\xi/3} \overline{\overline{S}}_4(\xi, \eta) e^{-\eta/3} \chi(\eta). \quad (3.98)$$

In the same way we get

$$\lim_{M \rightarrow \infty} \chi(\xi) e^{\xi/3} \widetilde{SD}_4(\xi, \eta) e^{\eta/3} \chi(\eta) = \chi(\xi) e^{\xi/3} \overline{\overline{SD}}_4(\xi, \eta) e^{\eta/3} \chi(\eta), \quad (3.99)$$

$$\lim_{M \rightarrow \infty} \chi(\xi) e^{-\xi/3} \widetilde{IS}_4(\xi, \eta) e^{-\eta/3} \chi(\eta) = \chi(\xi) e^{-\xi/3} \overline{\overline{IS}}_4(\xi, \eta) e^{-\eta/3} \chi(\eta). \quad (3.100)$$

Then we get the convergence in trace norm of a conjugate of $\chi(\xi)\tilde{P}_T(\xi, \eta)\chi(\eta)$

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \chi(\xi) \begin{pmatrix} e^{\xi/3} \tilde{S}_4(\xi, \eta) e^{-\eta/3} & e^{\xi/3} \widetilde{SD}_4(\xi, \eta) e^{\eta/3} \\ e^{-\xi/3} \widetilde{IS}_4(\xi, \eta) e^{-\eta/3} & e^{-\xi/3} \tilde{S}_4(\eta, \xi) e^{\eta/3} \end{pmatrix} \chi(\eta) \\
&= \chi(\xi) \begin{pmatrix} e^{\xi/3} \bar{\bar{S}}_4(\xi, \eta) e^{-\eta/3} & e^{\xi/3} \bar{\bar{SD}}_4(\xi, \eta) e^{\eta/3} \\ e^{-\xi/3} \bar{\bar{IS}}_4(\xi, \eta) e^{-\eta/3} & e^{-\xi/3} \bar{\bar{S}}_4(\eta, \xi) e^{\eta/3} \end{pmatrix} \chi(\eta) \\
&= \chi(\xi) \begin{pmatrix} e^{\xi/3} & 0 \\ 0 & e^{-\xi/3} \end{pmatrix} \begin{pmatrix} \bar{\bar{S}}_4(\xi, \eta) & \bar{\bar{SD}}_4(\xi, \eta) \\ \bar{\bar{IS}}_4(\xi, \eta) & \bar{\bar{S}}_4(\eta, \xi) \end{pmatrix} \begin{pmatrix} e^{-\eta/3} & 0 \\ 0 & e^{\eta/3} \end{pmatrix} \chi(\eta).
\end{aligned} \tag{3.101}$$

Therefore,

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \det(I - \tilde{P}_T(\xi, \eta)) \\
&= \det \left(I - \chi(\xi) \begin{pmatrix} e^{\xi/3} \bar{\bar{S}}_4(\xi, \eta) e^{-\eta/3} & e^{\xi/3} \bar{\bar{SD}}_4(\xi, \eta) e^{\eta/3} \\ e^{-\xi/3} \bar{\bar{IS}}_4(\xi, \eta) e^{-\eta/3} & e^{-\xi/3} \bar{\bar{S}}_4(\eta, \xi) e^{\eta/3} \end{pmatrix} \chi(\eta) \right) \\
&= \det \left(I - \chi(\xi) \begin{pmatrix} \bar{\bar{S}}_4(\xi, \eta) & \bar{\bar{SD}}_4(\xi, \eta) \\ \bar{\bar{IS}}_4(\xi, \eta) & \bar{\bar{S}}_4(\eta, \xi) \end{pmatrix} \chi(\eta) \right),
\end{aligned} \tag{3.102}$$

we use the matrix version of proposition 2.6.

3.4.3 The $a > \gamma^{-1}$ part

If $a > \gamma^{-1}$, the location as well as the fluctuation scale of the largest sample eigenvalue is changed. We change variables as $p = (a+1) \left(1 + \frac{1}{\gamma^2 a}\right)$ and $q = (a+1) \sqrt{1 - \frac{1}{\gamma^2 a^2} \frac{1}{\sqrt{2M}}}$, and then by (3.72) the kernel $S_*(x, y)$ after substitution is ³

$$\begin{aligned}
\tilde{S}_*(\xi, \eta) = (a+1) \sqrt{1 - \frac{1}{\gamma^2 a^2} \frac{1}{\sqrt{2M}}} S_*(x, y) \Bigg|_{\substack{x=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}}\xi \\ y=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}}\eta}
\end{aligned} \tag{3.103}$$

³Here $*$ stands for 4, $4a$ or $4b$, and the $\tilde{S}_*(\xi, \eta)$ in this subsection is not identical to that in subsection 3.4.1 and 3.4.2.

We analyze $\tilde{S}_{4b}(\xi, \eta)$ first. As before, we have L^2 convergence results by (5.185) and similar results

$$\begin{aligned} & \lim_{M \rightarrow \infty} \frac{(\gamma^2 a + 1)^{M-N+1/2}}{(\gamma^2 a)^{M+N+1/2}(a+1)^{M-N-1/2}} \sqrt{(\gamma^2 a^2 - 1)2M} e^{M-N} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx} \\ & e^{2\xi/3} L_{2N-1}^{(2(M-N))} (2Mx) x^{M-N-1/2} e^{-Mx} \chi(\xi) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4} \frac{\gamma^4 a^2 + \gamma^2 a^2 + 4\gamma^2 a + \gamma^2 + 1}{(\gamma^2 a + 1)^2} \xi^2 + 2\xi/3} \chi(\xi), \end{aligned} \quad (3.104)$$

$$\begin{aligned} & \lim_{M \rightarrow \infty} \frac{1}{2} \frac{(\gamma^2 a + 1)^{M-N-1/2} (\gamma^2 a^2 - 1)}{(\gamma^2 a)^{M+N+1/2} (a+1)^{M-N+1/2}} \sqrt{\gamma^2 a^2 - 1} (2M)^{3/2} e^{M-N} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} My} \\ & e^{2\eta/3} \int_y^\infty L_{2N-1}^{(2(M-N))} (2Mt) t^{M-N-1/2} e^{-Mt} dt \chi(\eta) = \\ & -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4} \frac{\gamma^4 a^2 + \gamma^2 a^2 + 4\gamma^2 a + \gamma^2 + 1}{(\gamma^2 a + 1)^2} \eta^2 + 2\eta/3} \chi(\eta), \end{aligned} \quad (3.105)$$

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left(\frac{\gamma^2 a}{(\gamma^2 a + 1)(a+1)} \right)^{M-N+1/2} e^{M-N} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} My} \\ & e^{-2\eta/3} \psi_{2N-1}(y) \chi(\eta) = e^{-\frac{1}{4} \frac{(\gamma^2 a^2 - 1)(\gamma^2 - 1)}{(\gamma^2 a + 1)^2} \eta^2 - 2\eta/3} \chi(\eta), \end{aligned} \quad (3.106)$$

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left(\frac{\gamma^2 a}{(\gamma^2 a + 1)(a+1)} \right)^{M-N-1/2} e^{M-N} \frac{\gamma^2 a}{(\gamma^2 a^2 - 1)M} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx} \\ & e^{-2\xi/3} \psi'_{2N-1}(x) \chi(\xi) = e^{-\frac{1}{4} \frac{(\gamma^2 a^2 - 1)(\gamma^2 - 1)}{(\gamma^2 a + 1)^2} \xi^2 - 2\xi/3} \chi(\xi). \end{aligned} \quad (3.107)$$

For notational simplicity, we denote functions on the left-hand sides of (3.104) – (3.107) by $F_1(\xi)$, $F_2(\eta)$, $F_3(\eta)$ and $F_4(\xi)$, and denote

$$c_M = (2M)^{2(M-N)} \frac{(2N-1)!}{(2M-1)!} e^{2(N-M)} \gamma^{4N-1} \quad (3.108)$$

By (3.86), we have

$$\lim_{M \rightarrow \infty} c_M = 1. \quad (3.109)$$

Then we get from (3.69), (3.103) and (3.104)—(3.107)

$$\begin{aligned} \widetilde{S}_{4b}(\xi, \eta) = -\frac{c_M}{2} & \left(e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) - 2(\xi - \eta)/3} F_1(\xi) F_3(\eta) \right. \\ & \left. + e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) + 2(\xi - \eta)/3} F_4(\xi) F_2(\eta) \right). \end{aligned} \quad (3.110)$$

If we define

$$SD_{4a}(x, y) = \sum_{j=0}^{N-2} \frac{1}{r_j} (\psi'_{2j}(x) \psi'_{2j+1}(y) - \psi'_{2j+1}(x) \psi'_{2j}(y)), \quad (3.111)$$

$$IS_{4a}(x, y) = \sum_{j=0}^{N-2} \frac{1}{r_j} (-\psi_{2j}(x) \psi_{2j+1}(y) + \psi_{2j+1}(x) \psi_{2j}(y)), \quad (3.112)$$

and

$$SD_{4b}(x, y) = \frac{1}{r_{N-1}} (\psi'_{2N-2}(x) \psi'_{2N-1}(y) - \psi'_{2N-1}(x) \psi'_{2N-2}(y)), \quad (3.113)$$

$$IS_{4b}(x, y) = \frac{1}{r_{N-1}} (-\psi_{2N-2}(x) \psi_{2N-1}(y) + \psi_{2N-1}(x) \psi_{2N-2}(y)), \quad (3.114)$$

and by (3.71) and (3.73) like (3.103) ⁴

$$\begin{aligned} \widetilde{SD}_*(\xi, \eta) = (a+1)^2 & \left(1 - \frac{1}{\gamma^2 a^2} \right) \frac{1}{2M} SD_*(x, y) \Big|_{\substack{x=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}\xi, \\ y=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}\eta}} \end{aligned} \quad (3.115)$$

$$\begin{aligned} \widetilde{IS}_*(\xi, \eta) = IS_*(x, y) \Big|_{\substack{x=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}\xi, \\ y=(a+1)\left(1+\frac{1}{\gamma^2 a}\right)+(a+1)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}\eta}} \end{aligned} \quad (3.116)$$

⁴* stands for 4, 4a or 4b.

then in the same way of (3.110), we have

$$\begin{aligned} \widetilde{SD}_{4b}(\xi, \eta) = \frac{c_M}{4} C_M \left(e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) - 2(\xi - \eta)/3} F_1(\xi) F_4(\eta) \right. \\ \left. - e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) + 2(\xi - \eta)/3} F_4(\xi) F_1(\eta) \right), \end{aligned} \quad (3.117)$$

$$\begin{aligned} \widetilde{IS}_{4b}(\xi, \eta) = \frac{c_M}{C_M} \left(e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) - 2(\xi - \eta)/3} F_2(\xi) F_3(\eta) \right. \\ \left. - e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} M(x-y) + 2(\xi - \eta)/3} F_3(\xi) F_2(\eta) \right), \end{aligned} \quad (3.118)$$

with

$$C_M = \frac{(\gamma^2 a^2 - 1)^{3/2} \sqrt{2M}}{a\gamma(\gamma^2 a + 1)}. \quad (3.119)$$

Now we write $\widetilde{P}_T(\xi, \eta)$ as the sum

$$\widetilde{P}_T(\xi, \eta) = \widetilde{P}_{Ta}(\xi, \eta) + \widetilde{P}_{Tb}(\xi, \eta), \quad (3.120)$$

with

$$\widetilde{P}_{Ta}(\xi, \eta) = \chi(\eta) \begin{pmatrix} \widetilde{S}_{4a}(\xi, \eta) & \widetilde{SD}_{4a}(\xi, \eta) \\ \widetilde{IS}_{4a}(\xi, \eta) & \widetilde{S}_{4a}(\eta, \xi) \end{pmatrix} \chi(\eta), \quad (3.121)$$

$$\widetilde{P}_{Tb}(\xi, \eta) = \chi(\eta) \begin{pmatrix} \widetilde{S}_{4b}(\xi, \eta) & \widetilde{SD}_{4b}(\xi, \eta) \\ \widetilde{IS}_{4b}(\xi, \eta) & \widetilde{S}_{4b}(\eta, \xi) \end{pmatrix} \chi(\eta). \quad (3.122)$$

If we denote

$$U(\xi) = \begin{pmatrix} e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx + 2\xi/3} & -\frac{C_M}{2} \frac{F_4(\xi)}{F_3(\xi)} e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx + 2\xi/3} \\ 0 & e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx - 2\xi/3} \end{pmatrix}, \quad (3.123)$$

$$U^{-1}(\eta) = \begin{pmatrix} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} My - 2\eta/3} & \frac{C_M}{2} \frac{F_4(\xi)}{F_3(\xi)} e^{-\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx - 2\xi/3} \\ 0 & e^{\frac{\gamma^2 a^2 - 1}{(\gamma^2 a + 1)(a+1)} Mx + 2\xi/3} \end{pmatrix}, \quad (3.124)$$

then we have the result of kernel conjugation

$$U(\xi)\tilde{P}_{Tb}(\xi, \eta)U^{-1}(\eta) = \chi(\xi) \begin{pmatrix} -\frac{c_M}{2} \left(F_1(\xi) + \frac{F_2(\xi)F_4(\xi)}{F_3(\xi)} \right) F_3(\eta) & 0 \\ U(\xi)\tilde{P}_{Tb}(\xi, \eta)U^{-1}(\eta)_{21} & -\frac{c_M}{2} F_3(\xi) \left(F_1(\eta) + \frac{F_2(\eta)F_4(\eta)}{F_3(\eta)} \right) \end{pmatrix} \chi(\eta), \quad (3.125)$$

with the entry

$$U(\xi)\tilde{P}_{Tb}(\xi, \eta)U^{-1}(\eta)_{21} = \frac{c_M}{C_M} \left(e^{-\frac{2(\gamma^2 a^2 - 1)}{(\gamma^2 a + 1)(a+1)} Mx - 4\xi/3} F_2(\xi)F_3(\eta) - F_3(\xi)F_2(\eta)e^{-\frac{2(\gamma^2 a^2 - 1)}{(\gamma^2 a + 1)(a+1)} My - 4\eta/3} \right). \quad (3.126)$$

We want $U(\xi)\tilde{P}_{Tb}(\xi, \eta)U^{-1}(\eta)$ to converge in trace norm as $M \rightarrow \infty$, and need the result

Lemma 3.2. *In L^2 norm,*

$$\lim_{M \rightarrow \infty} \frac{F_2(\xi)F_4(\xi)}{F_3(\xi)} \chi(\xi) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4} \frac{\gamma^4 a^2 + \gamma^2 a^2 + 4\gamma^2 a + \gamma^2 + 1}{(\gamma^2 a + 1)^2} \xi^2 + 2\xi/3} \chi(\xi). \quad (3.127)$$

The proof is left to the reader. The main ingredient is (3.105) and the fact that $F_4(\xi)/F_3(\xi)$ approaches to 1 uniformly on $[T, \infty)$.

We need another convergence result on $U(\xi)\tilde{P}_{Ta}(\xi, \eta)U^{-1}(\eta)$:

Proposition 3.1. *In trace norm,*

$$\lim_{M \rightarrow \infty} U(\xi)\tilde{P}_{Ta}(\xi, \eta)U^{-1}(\eta) = 0. \quad (3.128)$$

The proof is left to the reader. Since all the four entries in $\tilde{P}_{Ta}(\xi, \eta)$ can be expressed by Laguerre polynomials, the asymptotic results like (3.104) and (3.105) give the convergence (3.128).

By lemma 3.2 and proposition 3.1, we get in trace norm

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \det(I - \tilde{P}_T(\xi, \eta)) \\
&= \lim_{M \rightarrow \infty} \det(I - U(\xi) \tilde{P}_T(\xi, \eta) U^{-1}(\eta)) \\
&= \lim_{M \rightarrow \infty} \det(I - U(\xi) \tilde{P}_{Tb}(\xi, \eta) U^{-1}(\eta)) \\
&= \left(\int_{-\infty}^T \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right)^2,
\end{aligned} \tag{3.129}$$

and we get the proof of the $a > \gamma^{-1}$ part of theorem 1.2.

Chapter 4

Phase transition phenomenon

4.1 Rank 1 complex spiked model

Here we assume that the single spiked population eigenvalue is $1 + a = 1 + \gamma^{-1}$, and by the part 2 of theorem 1.1, we know the distribution function of the largest sample eigenvalue is $F_{\text{GUE}1}$, which is, according to (1.67), defined as

$$F_{\text{GUE}1}(T) = \det \left(1 - \chi(\xi) \left(K_{\text{Airy}}(\xi, \eta) + \text{Ai}(\xi) s^{(1)}(\eta) \right) \chi(\eta) \right). \quad (4.1)$$

Forrester recognized that [10]

$$F_{\text{GUE}1}(T) = F_{\text{GOE}}^2(T). \quad (4.2)$$

Therefore as the perturbative parameter a increases, by (1.70) and (4.2) we have the $F_{\text{GUE}} - F_{\text{GOE}}^2$ —Gaussian phase transition phenomenon around $a = \gamma^{-1}$.

4.2 Rank 1 quaternionic spiked model

Again we assume that the single spiked population eigenvalue is $1 + a = 1 + \gamma^{-1}$. As the perturbative parameter a increases, we have the $F_{\text{GSE}} - F_{\text{GOE}}$ —Gaussian phase transition phenomenon around $a = \gamma^{-1}$, by results of theorem 1.2 and 1.3. In this section we prove theorem 1.3.

In manipulation of kernels, we follow the method of [29]. The procedure seems informal and cursory, but is carefully justified in [29].

For notational simplicity, we denote $(\chi(\xi) = \chi_{(T, \infty)}(\xi))$

$$B(\xi) = 1 - s^{(1)}(\xi) = \int_{\xi}^{\infty} \text{Ai}(t) dt. \quad (4.3)$$

First, we express the integral operator

$$\chi(\xi) \overline{\overline{P}}(\xi, \eta) \chi(\eta) = \begin{pmatrix} \chi(\xi) \overline{\overline{S}}_4(\xi, \eta) \chi(\eta) & \chi(\xi) \overline{\overline{SD}}_4(\xi, \eta) \chi(\eta) \\ \chi(\xi) \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \chi(\xi) \overline{\overline{S}}_4(\eta, \xi,) \chi(\eta) \end{pmatrix} \quad (4.4)$$

by

$$\begin{pmatrix} \chi(\xi) \frac{\partial}{\partial \xi} & 0 \\ 0 & \chi(\xi) \end{pmatrix} \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \\ \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi,) \chi(\eta) \end{pmatrix}, \quad (4.5)$$

since by (3.34) – (3.37) and taking limit,

$$\frac{\partial}{\partial \xi} \overline{\overline{IS}}_4(\xi, \eta) = \overline{\overline{S}}_4(\xi, \eta), \quad (4.6)$$

$$\frac{\partial}{\partial \xi} \overline{\overline{S}}_4(\eta, \xi) = \overline{\overline{SD}}_4(\xi, \eta). \quad (4.7)$$

Then using (3.29) for A bounded and B trace class, upon suitably defining the Hilbert

spaces our operators A and B are acting on, we find

$$\begin{aligned}
& \det \left(I - \begin{pmatrix} \chi(\xi) \frac{\partial}{\partial \xi} & 0 \\ 0 & \chi(\xi) \end{pmatrix} \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \\ \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi,) \chi(\eta) \end{pmatrix} \right) \\
&= \det \left(I - \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \\ \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi,) \chi(\eta) \end{pmatrix} \begin{pmatrix} \chi(\eta) \frac{\partial}{\partial \eta} & 0 \\ 0 & \chi(\eta) \end{pmatrix} \right) \\
&= \det \left(I - \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} & \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \\ \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} & \overline{\overline{S}}_4(\eta, \xi,) \chi(\eta) \end{pmatrix} \right),
\end{aligned}$$

and by conjugation with $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, we get

$$\begin{aligned}
&= \det \left(I - \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} + \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) & \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \\ 0 & 0 \end{pmatrix} \right) \\
&= \det \left(I - \begin{pmatrix} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} + \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \end{pmatrix} \right). \tag{4.8}
\end{aligned}$$

Since

$$\int_T^\infty \overline{\overline{IS}}_4(\xi, \eta) \frac{\partial}{\partial \eta} f(\eta) d\eta = \overline{\overline{IS}}_4(\xi, \eta) f(\eta) \Big|_{\eta=T}^{\eta=\infty} - \int_T^\infty \frac{\partial}{\partial \eta} \overline{\overline{IS}}_4(\xi, \eta) f(\eta) d\eta, \tag{4.9}$$

as an operator

$$\overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} = \overline{\overline{IS}}_4(\xi, \infty) \delta_\infty(\eta) - \overline{\overline{IS}}_4(\xi, T) \delta_T(\eta) - \frac{\partial}{\partial \eta} \overline{\overline{IS}}_4(\xi, \eta) \chi(\eta), \tag{4.10}$$

where δ_∞ and δ_T are (generalized) Dirac functions. Then with the help of identity

$$\int_\xi^\infty K_{\text{Airy}}(t, \eta) dt + \int_\eta^\infty K_{\text{Airy}}(\xi, t) dt = \int_\xi^\infty \text{Ai}(t) dt \int_\xi^\infty \text{Ai}(t) dt, \tag{4.11}$$

which can be proved directly from (1.61), we get

$$\begin{aligned}
& I - \left(\overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} + \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \right) \\
&= I - \left(K_{\text{Airy}}(\xi, \eta) - \frac{1}{2} B(\xi) \text{Ai}(\eta) + \text{Ai}(\eta) \right) \chi(\eta) \\
&\quad + \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4} B(T) B(\xi) - \frac{1}{2} B(\xi) + \frac{1}{2} B(T) \right) \delta_T(\eta) \\
&\quad + \frac{1}{2} B(\xi) \delta_\infty(\eta).
\end{aligned} \tag{4.12}$$

Now we denote $R(\xi, \eta)$ as the resolvent of $K_{\text{Airy}}(\xi, \eta) \chi(\eta)$, such that as integral operators

$$I + R(\xi, \eta) = (I - K_{\text{Airy}}(\xi, \eta) \chi(\eta))^{-1}, \tag{4.13}$$

then

$$\begin{aligned}
& I - \left(\overline{\overline{IS}}_4(\xi, \eta) \chi(\eta) \frac{\partial}{\partial \eta} + \overline{\overline{S}}_4(\eta, \xi) \chi(\eta) \right) \\
&= (I - K_{\text{Airy}}(\xi, \eta) \chi(\eta)) \left(I - (I + R)(1 - \frac{1}{2} B(\xi)) \text{Ai}(\eta) \chi(\eta) \right. \\
&\quad + (I + R) \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4} B(T) B(\xi) - \frac{1}{2} B(\xi) + \frac{1}{2} B(T) \right) \delta_T(\eta) \\
&\quad \left. + \frac{1}{2} (I + R) B(\xi) \delta_\infty(\eta) \right).
\end{aligned} \tag{4.14}$$

Again by the formula (3.29), in the form of (formula (17) in [29])

$$\det \left(I - \sum_{k=1}^n \alpha_k \otimes \beta_k \right) = \det (\delta_{j,k} - (\alpha_j, \beta_k))_{j,k=1,\dots,n} \tag{4.15}$$

we get

$$\begin{aligned}
& \det \left(I - (I + R) \left(1 - \frac{1}{2} B(\xi) \right) \text{Ai}(\eta) \chi(\eta) \right. \\
& \quad + (I + R) \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4} B(T) B(\xi) - \frac{1}{2} B(\xi) + \frac{1}{2} B(T) \right) \delta_T(\eta) \\
& \quad \left. + \frac{1}{2} (I + R) B(\xi) \delta_\infty(\eta) \right) \\
& = \det \begin{pmatrix} 1 + \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1 + \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1 + \alpha_{33} \end{pmatrix},
\end{aligned} \tag{4.16}$$

where upon the definition

$$\langle f(\xi), g(\xi) \rangle_T = \int_T^\infty f(\xi) g(\xi) d\xi, \tag{4.17}$$

we have

$$\alpha_{11} = \left\langle (I + R)(1 - \frac{1}{2}B(\xi)), -\text{Ai}(\xi) \right\rangle_T, \quad (4.18)$$

$$\alpha_{12} = \left\langle (I + R) \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4}B(T)B(\xi) - \frac{1}{2}B(\xi) + \frac{1}{2}B(T) \right), -\text{Ai}(\xi) \right\rangle_T, \quad (4.19)$$

$$\alpha_{13} = \left\langle \frac{1}{2}(I + R)B(\xi), -\text{Ai}(\xi) \right\rangle_T, \quad (4.20)$$

$$\alpha_{21} = (I + R)(1 - \frac{1}{2}B(\xi)) \Big|_{\xi=T}, \quad (4.21)$$

$$\alpha_{22} = (I + R) \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4}B(T)B(\xi) - \frac{1}{2}B(\xi) + \frac{1}{2}B(T) \right) \Big|_{\xi=T}, \quad (4.22)$$

$$\alpha_{23} = \frac{1}{2}(I + R)B(\xi) \Big|_{\xi=T}, \quad (4.23)$$

$$\alpha_{31} = (I + R)(1 - \frac{1}{2}B(\xi)) \Big|_{\xi=\infty} = 1, \quad (4.24)$$

$$\begin{aligned} \alpha_{32} &= (I + R) \left(\frac{1}{2} \int_T^\infty K_{\text{Airy}}(\xi, t) dt - \frac{1}{4}B(T)B(\xi) - \frac{1}{2}B(\xi) + \frac{1}{2}B(T) \right) \Big|_{\xi=\infty} \\ &= \frac{1}{2}B(T). \end{aligned} \quad (4.25)$$

$$\alpha_{33} = \frac{1}{2}(I + R)B(\xi) \Big|_{\xi=\infty} = 0, \quad (4.26)$$

If we take elementary row operations, we get

$$\begin{aligned} & \det \begin{pmatrix} 1 + \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1 + \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1 + \alpha_{33} \end{pmatrix} \\ &= \det \begin{pmatrix} 1 + \alpha_{11} - \alpha_{13} & \alpha_{12} - \frac{1}{2}B(T)\alpha_{13} & \alpha_{13} \\ \alpha_{21} - \alpha_{23} & 1 + \alpha_{22} - \frac{1}{2}B(T)\alpha_{23} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 + \beta_{11} & \beta_{12} \\ \beta_{21} & 1 + \beta_{22} \end{pmatrix}, \end{aligned} \quad (4.27)$$

where

$$\beta_{11} = \langle (I + R)(1 - B(\xi)), -\text{Ai}(\xi) \rangle_T, \quad (4.28)$$

$$\beta_{12} = \left\langle \frac{1}{2}(I + R) \left(\int_T^\infty K_{\text{Airy}}(\xi, t) dt - B(T)B(\xi) - B(\xi) + B(T) \right), -\text{Ai}(\xi) \right\rangle_T, \quad (4.29)$$

$$\beta_{21} = (I + R)(1 - B(\xi))|_{\xi=T}, \quad (4.30)$$

$$\beta_{22} = \frac{1}{2}(I + R) \left(\int_T^\infty K_{\text{Airy}}(\xi, t) dt - B(T)B(\xi) - B(\xi) + B(T) \right) \Big|_{\xi=T}. \quad (4.31)$$

Using (4.13) and (4.15), we observe $(s^{(1)}(\xi) = 1 - B(\xi))$

$$\begin{aligned} & \det(I - K_{\text{Airy}}(\xi, \eta)\chi(\eta)) \det \begin{pmatrix} 1 + \beta_{11} & \beta_{12} \\ \beta_{21} & 1 + \beta_{22} \end{pmatrix} \\ &= \det \left(I - (K_{\text{Airy}}(\xi, \eta)\chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta))\chi(\eta) \right. \\ & \quad \left. + \frac{1}{2} \left(\int_T^\infty K_{\text{Airy}}(\xi, t) dt - B(T)B(\xi) - B(\xi) + B(T) \right) \delta_T(\eta) \right). \end{aligned} \quad (4.32)$$

If we denote $\tilde{R}(\xi, \eta)$ as the resolvent of $(K_{\text{Airy}}(\xi, \eta)\chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta))\chi(\eta)$, so that as operators

$$I + \tilde{R}(\xi, \eta) = (I + (K_{\text{Airy}}(\xi, \eta)\chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta))\chi(\eta))^{-1}, \quad (4.33)$$

and

$$Q(\xi) = (I + \tilde{R}) \left(\int_T^\infty K_{\text{Airy}}(\xi, t) dt - B(T)B(\xi) - B(\xi) + B(T) \right), \quad (4.34)$$

then

$$F_{\text{GSE1}} = \det \left(I - (K_{\text{Airy}}(\xi, \eta)\chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta))\chi(\eta) \right) \det \left(I + \frac{1}{2}Q(\xi)\delta_T(\eta) \right). \quad (4.35)$$

To prove theorem 1.3, we need only (4.1) and (4.2) with ξ and η swapped, and

$$\det \left(I + \frac{1}{2} Q(\xi) \delta_T(\eta) \right) = 1, \quad (4.36)$$

which by (4.15) is equivalent to

$$Q(T) = 0. \quad (4.37)$$

If we take $f(\xi) = Q(\xi) + 1$, then (4.37) is

$$\begin{aligned} (I - (K_{\text{Airy}}(\xi, \eta) \chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta)) \chi(\eta)) (f(\xi) - 1) = \\ \int_T^\infty K_{\text{Airy}}(\xi, t) dt - B(T)B(\xi) - B(\xi) + B(T), \end{aligned} \quad (4.38)$$

which is equivalent to

$$(I - (K_{\text{Airy}}(\xi, \eta) \chi(\eta) + s^{(1)}(\xi) \text{Ai}(\eta)) \chi(\eta)) f(\xi) = s^{(1)}(\xi). \quad (4.39)$$

The integral equation (4.39) is solvable, and the solution is

$$f(\xi) = \frac{(I + R)s^{(1)}(\xi)}{1 - \langle (I + R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T}. \quad (4.40)$$

Therefore to prove the theorem (1.3) we need only to prove $f(T) = 1$, which is equivalent to

$$(I + R)s^{(1)}(T) = 1 - \langle (I + R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T. \quad (4.41)$$

This is a nontrivial result, but it can be derived by results in [29], with ¹

$$(I + R)s^{(1)}(T) = e^{-\int_T^\infty q(s) ds}, \quad (4.42)$$

$$\langle (I + R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T = 1 - e^{-\int_T^\infty q(s) ds}, \quad (4.43)$$

¹In section VII of [29] Tracy and Widom define function \bar{q} and \bar{u} for both GOE and GSE. Our $(I + R)s^{(1)}(T)$ is equal to $\sqrt{2}$ times their \bar{q} in GOE and our $\langle (I + R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T$ is equal to 2 times their \bar{u} in GOE.

where q is the Painlevé II function described in (1.32) and (1.33)

We can give a proof of (4.42) and (4.43), based on the method and results in [28].

First, assume T is fixed, then $(I + R)s^{(1)}$ is a function, and we have

$$\frac{d}{d\xi}(I + R)s^{(1)}(\xi) = (I + R)\frac{ds^{(1)}(\xi)}{d\xi} + \left[\frac{d}{d\xi}, (1 + R)\right]s^{(1)}(\xi). \quad (4.44)$$

Since $\frac{d}{d\xi}s^{(1)}(\xi) = \text{Ai}(\xi)$ and we have (2.13) in [28], which is

$$\left[\frac{d}{d\xi}, (1 + R)\right] = -(2 + R)\text{Ai}(\xi) \cdot (1 - K^t)^{-1}(\text{Ai}(\eta)\chi(\eta)) + R(\eta, T) \cdot \rho(T, \eta), \quad (4.45)$$

where $\rho(x, y) = \delta(x - y) + R(x, y)$ is the distribution kernel of $1 + R$, and K^t is the transpose (as an operator) of $K_{\text{Airy}}(\xi, \eta)\chi(\eta)$, we have

$$\begin{aligned} \frac{d}{d\xi}(I + R)s^{(1)}(\xi) &= (1 + R)\text{Ai}(\xi) \\ &\quad - (1 + R)\text{Ai}(\xi) \cdot \langle (I + R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T + R(\xi, T) \cdot (1 + R)s^{(1)}(T). \end{aligned} \quad (4.46)$$

If we regard T as a parameter, then we have

$$\frac{d}{dT}(I + R)s^{(1)}(\xi; T) = -R(\xi, T) \cdot (1 + R)s^{(1)}(T), \quad (4.47)$$

because (2.16) in [28] gives

$$\frac{1}{dT}(1 + R) = R(\xi, T) \cdot \rho(T, \eta). \quad (4.48)$$

Therefore, if we set $\xi = T$ and take the derivative with respect to the parameter T ,

we have

$$\begin{aligned} \frac{d}{dT}((1+R)s^{(1)}(T)) &= \left(\frac{d}{d\xi} + \frac{d}{dT} \right) ((1+R)s^{(1)}(T)) \Big|_{\xi=T} \\ &= (1+R) \text{Ai}(T) \cdot (1 - \langle (I+R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T). \end{aligned} \quad (4.49)$$

On the other hand, by (4.47) we have

$$\begin{aligned} \frac{d}{dT} \langle (I+R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T &= - (1+R)s^{(1)}(T) \cdot \text{Ai}(T) + \left\langle \frac{d}{dT} (I+R)s^{(1)}(\xi), \text{Ai}(\xi) \right\rangle_T \\ &= - (1+R)s^{(1)}(T) \cdot \left(\text{Ai}(T) + \int_T^\infty R(\xi, T) \text{Ai}(\xi) d\xi \right) \\ &= - (1+R)s^{(1)}(T) \cdot (1+R) \text{Ai}(T). \end{aligned} \quad (4.50)$$

(1.11) and (1.12) in [28] give the result

$$(1+R) \text{Ai}(T) = q(T), \quad (4.51)$$

and now we if we denote $(I+R)s^{(1)}(T) = s_T$ and $\langle (I+R)s^{(1)}(\xi), \text{Ai}(\xi) \rangle_T = w_T$, we have

$$\begin{cases} \frac{d}{dT} s_T = q(1 - w_T) \\ \frac{d}{dT} (1 - w_T) = q s_T. \end{cases} \quad (4.52)$$

Now we can get (4.42) and (4.43) by boundary conditions.

4.3 Conjectures of phase transition in quaternionic and real spiked models

In the complex spiked model, we have more complicated phase transition phenomenon for the limiting distribution of the largest sample eigenvalue, if the rank is greater

than 1. For example, if there are two spiked population eigenvalues $1 + \alpha_1$ and $1 + \alpha_2$, we have the phase diagram as α_1 and α_2 vary from -1 to ∞ :

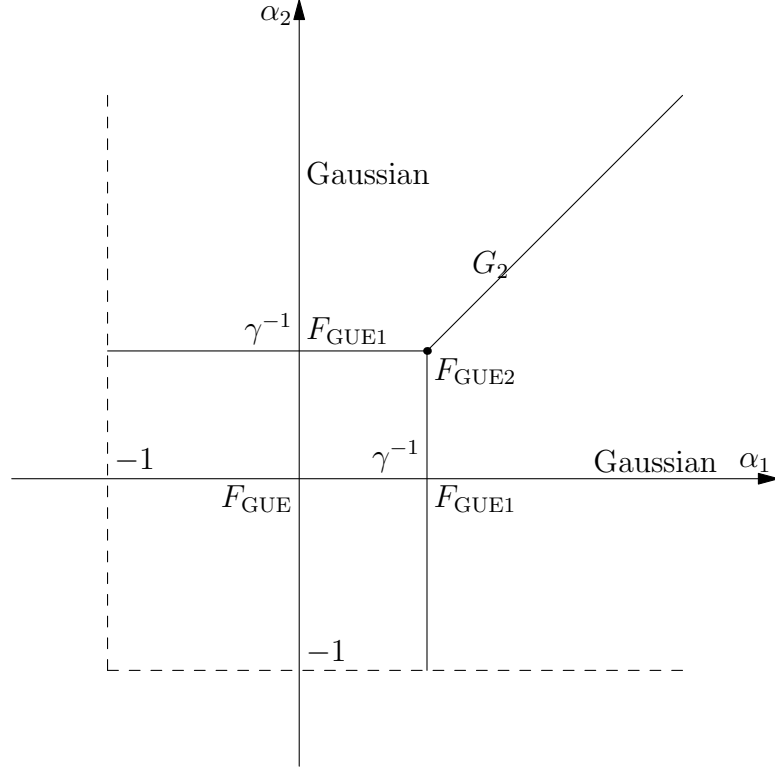
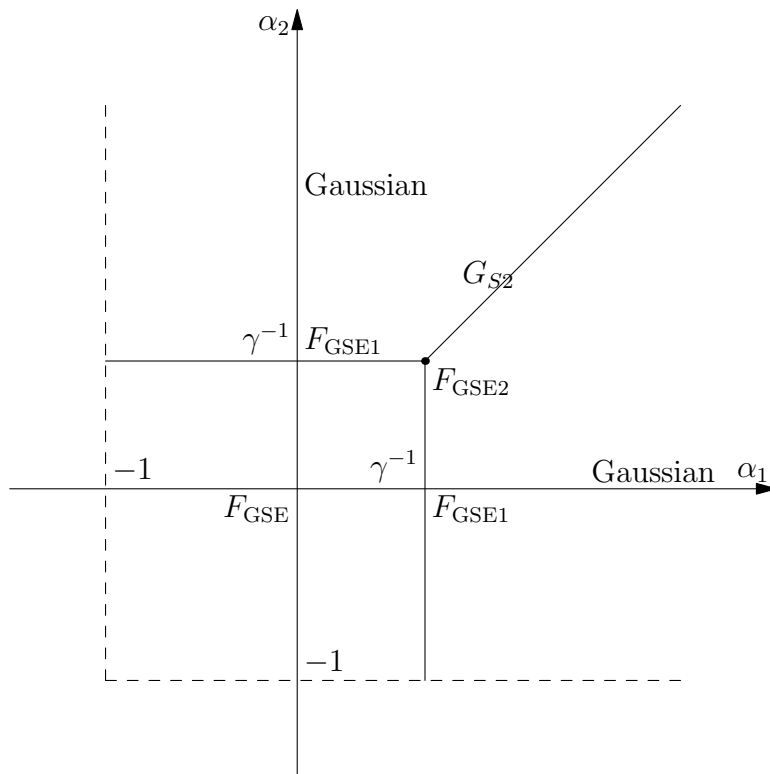


Figure 4.1: Phase diagram of rank 2 complex spiked model

Analogously, we conjecture the phase diagram for the rank 2 quaternionic spiked model for the limiting distribution of the largest sample eigenvalue, with spiked population eigenvalues $1 + \alpha_1$ and $1 + \alpha_2$. The G_{S1} is the distribution function of the largest eigenvalue of a 2×2 random quaternionic Hermitian matrix $\begin{pmatrix} a & c+id+je+kf \\ c-id-je-kf & b \end{pmatrix}$, where a, \dots, f are independent normal random variables with mean 0, the variance of a and b is 1, and the variance of c, \dots, f is $1/2$. Our conjecture for G_{S2} is based on the pattern of G_t for the rank t complex spiked model: Actually G_t is the distribution function of the largest eigenvalue of a $t \times t$ random Hermitian matrix $(a_{ij})_{1 \leq i, j \leq t}$, where a_{ii} , $\Re(a_{ij})$, $\Im(a_{ij})$ ($i < j$) are independent normal random variables with mean 0, the variance of diagonal entries is 1, and the variance of real and imaginary parts



of off-diagonal entries is $1/2$. Therefore we guess such distribution function for rank 2 quaternionic spiked model should be the distribution function of the largest eigenvalue of a $t \times t$ random Hermitian matrix. It is obvious that in the rank 1 case, the Gaussian distribution satisfies the conjecture trivially, and the next distribution function is G_{S_2} . But what is F_{GSE_2} ? Is $F_{\text{GSE}_2}^2$ a Fredholm determinant? Our only clue is that the F_{GSE_2} should be similar to the F_0^{semi} defined in [24].

For the real spiked model, even the rank 1 case is speculative. We conjecture that if the only spiked population eigenvalue is $1 + a$, then the limiting distribution of the largest sample eigenvalue has the pattern $F_{\text{GOE}} \text{---} F_{\text{GSE}} \text{---} \text{Gaussian}$. The Gaussian part has been proved in [23], and other results are missing.

Chapter 5

Asymptotic analysis

In sections 5.1–5.3, the convention of notations is the same as that in chapter 2, e.g., $\psi_{r'}$ is defined by (2.77). In section 5.4, the convention of notations is the same as that in chapter 3, e.g., $\psi_{r'}$ is defined by (3.19).

5.1 Asymptotics of $\psi(p + q\xi)$, $\psi_{r'}(p + q\xi)$, $\tilde{\psi}(p + q\eta)$ and $\tilde{\psi}_{r'}(p + q\eta)$ when $a_{s'} < \gamma^{-1}$

In this section, we assume $x = p + q\xi$ and $y = p + q\eta$, where $p = (1 + \gamma^{-1})^2$ and $q = \frac{(1+\gamma)^{4/3}}{\gamma M^{2/3}}$.

For the asymptotic analysis, we define $\bar{\Sigma}^\infty = \bar{\Sigma}_1^\infty \cup \bar{\Sigma}_2^\infty \cup \bar{\Sigma}_3^\infty$, where

$$\bar{\Sigma}_1^\infty = \{te^{\frac{2\pi i}{3}} \mid t \geq 1\}, \quad (5.1)$$

$$\bar{\Sigma}_2^\infty = \{e^{-t\pi i} \mid -\frac{4}{3} \leq t \leq -\frac{2}{3}\}, \quad (5.2)$$

$$\bar{\Sigma}_3^\infty = \{-te^{\frac{4\pi i}{3}} \mid t \leq -1\}, \quad (5.3)$$

and ($c < 0$)

$$\bar{\Sigma}_{>c}^\infty = \{w \in \bar{\Sigma}^\infty \mid \Re(w) > c\}, \quad (5.4)$$

$$\bar{\Sigma}_{\leq c}^\infty = \bar{\Sigma}^\infty \setminus \bar{\Sigma}_{>c}^\infty. \quad (5.5)$$

We have

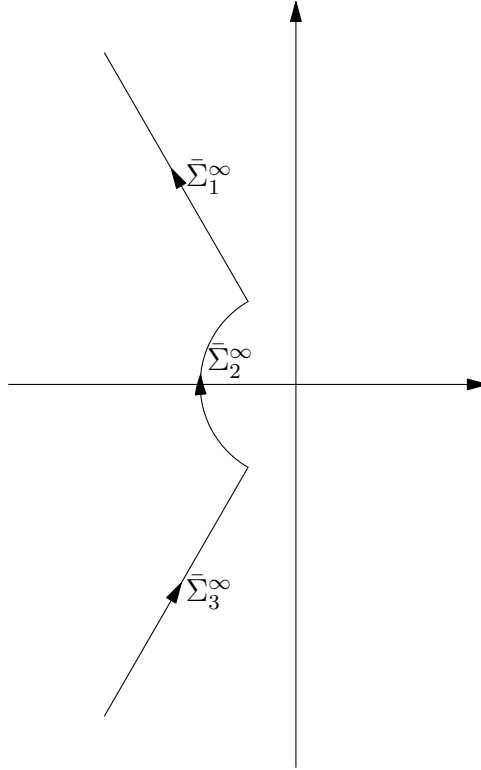


Figure 5.1: $\bar{\Sigma}^\infty$

$$\frac{1}{2\pi i} \int_{\bar{\Sigma}^\infty} e^{\xi u - \frac{u^3}{3}} du = \text{Ai}(\xi), \quad (5.6)$$

since by the substitution $u = iv$, we get ($\int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}}$ is defined below 1.62)

$$\frac{1}{2\pi i} \int_{\bar{\Sigma}^\infty} e^{\xi u - \frac{u^3}{3}} du = \frac{1}{2\pi} \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} e^{i\xi v + \frac{iv^3}{3}} dv, \quad (5.7)$$

which agrees with the integral definition of the Airy function, with the integration on the right hand side from $\infty e^{5\pi i/6}$ to $\infty e^{\pi i/6}$. By direct calculation, we also have the

result that for any T , if $-c$ is large enough

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_{\leq c}^{\infty}} e^{Tu - \frac{u^3}{3}} du \right| < \frac{1}{-c}. \quad (5.8)$$

Similarly, we define $\bar{\Gamma}^{\infty} = \bar{\Gamma}_1^{\infty} \cup \bar{\Gamma}_2^{\infty} \cup \bar{\Gamma}_3^{\infty}$, where

$$\bar{\Gamma}_1^{\infty} = \{-te^{\frac{\pi i}{3}} \mid t \leq -1\}, \quad (5.9)$$

$$\bar{\Gamma}_2^{\infty} = \{e^{-t\pi i} \mid -\frac{1}{3} \leq t \leq \frac{1}{3}\}, \quad (5.10)$$

$$\bar{\Gamma}_3^{\infty} = \{te^{\frac{5\pi i}{3}} \mid t \geq 1\}, \quad (5.11)$$

and

$$\bar{\Gamma}_{<c}^{\infty} = \{w \in \bar{\Gamma}^{\infty} \mid \Re(w) < c\}, \quad (5.12)$$

$$\bar{\Gamma}_{\geq c}^{\infty} = \bar{\Gamma}^{\infty} \setminus \bar{\Sigma}_{<c}^{\infty}. \quad (5.13)$$

We have (similar to (5.6))

$$\frac{1}{2\pi i} \int_{\bar{\Gamma}^{\infty}} e^{-\eta u + \frac{u^3}{3}} du = -\text{Ai}(\xi), \quad (5.14)$$

and for any T , if c is large enough

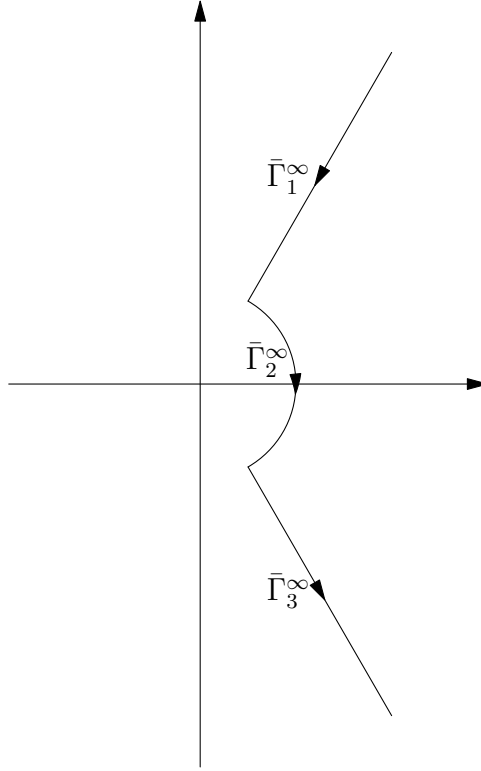
$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}_{\geq c}^{\infty}} e^{-Tu + \frac{u^3}{3}} du \right| < \frac{1}{c}. \quad (5.15)$$

5.1.1 Asymptotics of $\psi(p + q\xi)$ and $\psi_{r'}(p + q\xi)$

We only analyze $\psi_{r'}(p + q\xi)$. The analysis of $\psi(p + q\xi)$ is similar and simpler, and we only give the result.

First we have

$$e^{Mxz} \frac{(z-1)^N}{z^M} = e^{-Mf(z) + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi z}, \quad (5.16)$$


 Figure 5.2: $\bar{\Gamma}^\infty$

where

$$f(z) = -(1 + \gamma^{-1})^2 z + \log z - \gamma^{-2} \log(z - 1), \quad (5.17)$$

and here and later, we do not need to concern ourselves about the ambiguity of the value of logarithmic functions. Now we can write (2.77) as

$$\begin{aligned} \psi_{r'}(x) &= \frac{1}{2\pi i} \\ &\oint_{\Sigma} e^{-Mf(z) + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi z} \frac{z^{r-r'}}{(z-1)^r} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{t'-1} dz. \end{aligned} \quad (5.18)$$

For $f(z)$ we have

- $f'(z) = \frac{((1 + \gamma^{-1})z - 1)^2}{z(z - 1)}$, with the zero point $z = \frac{\gamma}{\gamma+1}$;

- $f''\left(\frac{\gamma}{\gamma+1}\right) = 0;$
- $f'''\left(\frac{\gamma}{\gamma+1}\right) = \frac{2(\gamma+1)^4}{\gamma^3} > 0.$

Hence locally around $z = \frac{\gamma}{\gamma+1},$

$$f\left(\frac{\gamma}{\gamma+1} + w\right) = -\frac{\gamma+1}{\gamma} + \log \gamma - (1-\gamma^2)\log(\gamma+1) + \gamma^{-2}\pi i + \frac{(\gamma+1)^4}{3\gamma^3}w^3 + R_1(w), \quad (5.19)$$

where

$$R_1(w) = \mathcal{O}(w^4), \quad \text{as } w \rightarrow 0. \quad (5.20)$$

After the substitution $z = w + \frac{\gamma}{\gamma+1},$ we get by (5.18)

$$\begin{aligned} & \psi_{r'}(p + q\xi) \\ &= \frac{1}{2\pi i} \oint_{\bar{\Sigma}^M} e^{M\left(\frac{\gamma+1}{\gamma} - \log \gamma + (1-\gamma^{-2})\log(\gamma+1) - \gamma^{-2}\pi i - \frac{(\gamma+1)^4}{3\gamma^3}w^3 - R_1(w)\right) + \frac{(1+\gamma)^{4/3}}{\gamma}M^{1/3}\xi\left(w + \frac{\gamma}{\gamma+1}\right)} \\ & \quad \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)}\right)^{r_j}\right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})}\right)^{t'-1} dw \\ &= \frac{(-1)^N}{2\pi i} \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1}Mx} \oint_{\bar{\Sigma}^M} e^{\frac{(1+\gamma)^{4/3}}{\gamma}M^{1/3}\xi w - \frac{(\gamma+1)^4}{3\gamma^3}Mw^3 - MR_1(w)} \\ & \quad \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)}\right)^{r_j}\right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})}\right)^{t'-1} dw, \end{aligned} \quad (5.21)$$

where $\bar{\Sigma}^M$ is a contour around $w = -\frac{\gamma}{\gamma+1},$ composed of $\bar{\Sigma}_1^M, \bar{\Sigma}_2^M, \bar{\Sigma}_3^M$ and $\bar{\Sigma}_4^M,$ which

are defined as

$$\bar{\Sigma}_1^M = \left\{ t \frac{\gamma}{\gamma+1} e^{\frac{2\pi i}{3}} \left| \frac{1}{(\gamma+1)^{1/3}} M^{-1/3} \leq t \leq 4 \right. \right\}, \quad (5.22)$$

$$\bar{\Sigma}_2^M = \left\{ \frac{\gamma}{(\gamma+1)^{4/3}} M^{-1/3} e^{-t\pi i} \left| -\frac{4}{3} \leq t \leq -\frac{2}{3} \right. \right\}, \quad (5.23)$$

$$\bar{\Sigma}_3^M = \left\{ (4-t) \frac{\gamma}{\gamma+1} e^{\frac{4\pi i}{3}} \left| 0 \leq t \leq 4 - \frac{1}{(\gamma+1)^{1/3}} M^{-1/3} \right. \right\}, \quad (5.24)$$

$$\bar{\Sigma}_4^M = \left\{ -2 \frac{\gamma}{\gamma+1} - it \left| -2\sqrt{3} \frac{\gamma}{(\gamma+1)} \leq t \leq 2\sqrt{3} \frac{\gamma}{(\gamma+1)} \right. \right\}. \quad (5.25)$$

For asymptotic analysis, we define

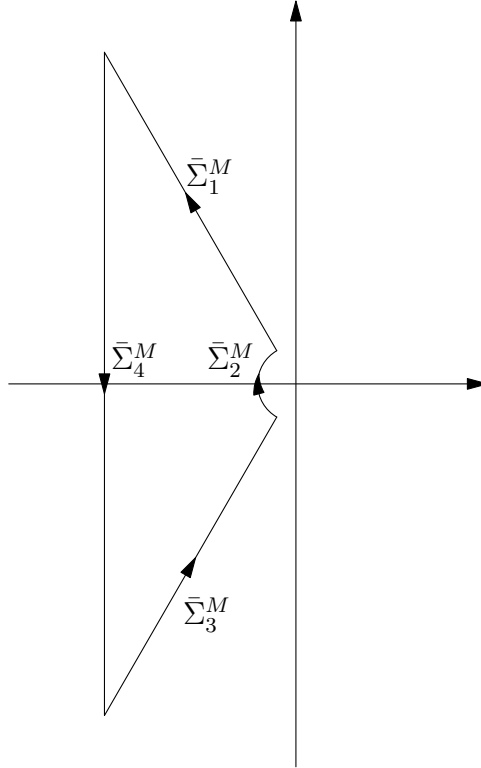


Figure 5.3: $\bar{\Sigma}^M$

$$\bar{\Sigma}_{\text{local}}^M = \{z \in \bar{\Sigma}^M \mid \Re(z) > -M^{-10/39}\}, \quad (5.26)$$

$$\bar{\Sigma}_{\text{remote}}^M = (\bar{\Sigma}_1^M \cup \bar{\Sigma}_3^M) \setminus \bar{\Sigma}_{\text{local}}^M. \quad (5.27)$$

Now if we denote

$$F_M(\xi, w) = \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} e^{\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi w - \frac{(\gamma+1)^4}{3\gamma^3} M w^3 - M R_1(w)} \\ \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_j)} \right)^{r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_{s'})} \right)^{t'-1}, \quad (5.28)$$

we have by (5.21)

$$\frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{\gamma^M}{(\gamma + 1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} M x} \psi_{r'}(p + q\xi) = \frac{1}{2\pi i} \oint_{\bar{\Sigma}^M} F_M(\xi, w) dw, \quad (5.29)$$

and establish several lemmas:

Lemma 5.1. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_4^M} F_M(\xi, w) dw \right| < \frac{1}{3} \frac{e^{-\xi/2}}{M^{1/40}}, \quad (5.30)$$

for any $\xi \geq T$.

Proof. By (5.17) and (5.19),

$$\begin{aligned} & \frac{(\gamma + 1)^4}{3\gamma^3} w^3 + R_1(w) \\ &= f\left(\frac{\gamma}{\gamma+1} + w\right) + \frac{\gamma}{\gamma+1} - \log \gamma + (1 - \gamma^{-2}) \log(\gamma + 1) - \gamma^{-2} \pi i \\ &= -\left(\frac{\gamma+1}{\gamma}\right)^2 w + \log\left(\frac{\gamma+1}{\gamma} w + 1\right) - \gamma^{-2} \log((\gamma+1)w - 1) - \gamma^{-2} \pi i. \end{aligned} \quad (5.31)$$

If $w \in \bar{\Sigma}_4^M$, $\Re(w) = -2\frac{\gamma}{\gamma+1}$, and denote $\theta = \arg(w) \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$, we have

$$\begin{aligned} & \Re \left(\frac{(\gamma+1)^4}{3\gamma^3} w^3 + R_1(w) \right) \\ &= 2\frac{\gamma}{\gamma+1} + \log \left(\sqrt{(-1)^2 + (2 \tan \theta)^2} \right) - \gamma^{-2} \log \left(\sqrt{(2\gamma+3)^2 + (2\gamma \tan \theta)^2} \right) \quad (5.32) \\ &\geq 2\frac{\gamma}{\gamma+1} + 0 - \gamma^{-2} \log \left(\sqrt{16\gamma^2 + 12\gamma + 9} \right), \end{aligned}$$

and we can prove that if $\gamma \geq 1$,

$$2\frac{\gamma}{\gamma+1} + 0 - \gamma^{-2} \log \left(\sqrt{16\gamma^2 + 12\gamma + 9} \right) > 2 - \log \sqrt{37} > 0. \quad (5.33)$$

Therefore on $\bar{\Sigma}_4^M$, if $\xi \geq T$, for $0 < \epsilon' < 2 - \log \sqrt{37}$ and M large enough,

$$\begin{aligned} & |F_M(\xi, w)| \\ &< \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} e^{-2(\xi-T)(\gamma+1)^{1/3} M^{1/3} + (\log \sqrt{37} - 2 + 2T(\gamma+1)^{1/3} M^{-2/3})M} \\ &\quad \times \left| \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})} \right)^{t'-1} \right| \\ &< e^{-2(\xi-T)(\gamma+1)^{1/3} M^{1/3}} e^{(\log \sqrt{37} - 2 + \epsilon')M}. \end{aligned} \quad (5.34)$$

If M is large enough,

$$e^{(\log \sqrt{37} - 2 + \epsilon')M} < \frac{2\pi}{4\sqrt{3}\frac{\gamma}{\gamma+1}} \frac{1}{3} e^{-T/2} M^{-1/40}, \quad (5.35)$$

$$e^{-2(\xi-T)(\gamma+1)^{1/3} M^{1/3}} < e^{T/2} e^{-\xi/2}, \quad (5.36)$$

and we get the result, since

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_4^M} F_M(\xi, w) dw \right| \leq \frac{4\sqrt{3}\frac{\gamma}{\gamma+1}}{2\pi} \max_{w \in \bar{\Sigma}_4^M} |F_M(\xi, w)| \quad (5.37)$$

□

Lemma 5.2. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\Sigma_{\text{remote}}^M} F_M(\xi, w) dw \right| < \frac{1}{3} \frac{e^{-\xi/2}}{M^{1/40}}, \quad (5.38)$$

for any $\xi \geq T$.

Proof. For $w \in \bar{\Sigma}_{\text{remote}}^M$, we denote $l = -\Re(w) = \frac{|w|}{2}$. Since $\arg(w) = \pm \frac{2\pi}{3}$, we get by (5.31)

$$\begin{aligned} \Re \left(\frac{(\gamma+1)^4}{3\gamma^3} w^3 + R_1(w) \right) &= \frac{(\gamma+1)^2}{\gamma^2} l \\ &+ \frac{1}{2} \log \left(1 - 2 \frac{\gamma+1}{\gamma} l + 4 \left(\frac{\gamma+1}{\gamma} l \right)^2 \right) - \frac{\gamma^{-2}}{2} \log (1 + 2(\gamma+1)l + 4(\gamma+1)^2 l^2). \end{aligned} \quad (5.39)$$

Then we take the derivative on both sides of (5.39)

$$\begin{aligned} \frac{d}{dl} \Re \left(\frac{(\gamma+1)^4}{3\gamma^3} w^3 + R_1(w) \right) &= \\ 8 \frac{(\gamma+1)^4}{\gamma^3} l^2 \frac{1 + (\gamma-1) \frac{\gamma+1}{\gamma} l + 2\gamma \left(\frac{\gamma+1}{\gamma} l \right)^2}{\left(1 - 2 \frac{\gamma+1}{\gamma} l + 4 \left(\frac{\gamma+1}{\gamma} l \right)^2 \right) (1 + 2(\gamma+1)l + 4(\gamma+1)^2 l^2)}, \end{aligned} \quad (5.40)$$

and are able to find a positive number ϵ'' , such that for $0 \leq l \leq 2 \frac{\gamma}{\gamma+1}$,

$$8 \frac{(\gamma+1)^4}{\gamma^3} l^2 \frac{1 + (\gamma-1) \frac{\gamma+1}{\gamma} l + 2\gamma \left(\frac{\gamma+1}{\gamma} l \right)^2}{\left(1 - 2 \frac{\gamma+1}{\gamma} l + 4 \left(\frac{\gamma+1}{\gamma} l \right)^2 \right) (1 + 2(\gamma+1)l + 4(\gamma+1)^2 l^2)} > 3\epsilon'' l^2, \quad (5.41)$$

and on the two right most points of $\bar{\Sigma}_{\text{remote}}^M$, $(-1 + \sqrt{3})M^{-10/39}$ and $(-1 - \sqrt{3})M^{-10/39}$,

$$\begin{aligned}
& \Re \left(\frac{(\gamma + 1)^4}{3\gamma^3} w^3 + R_1(w) \right) \Big|_{w=(-1 \pm \sqrt{3})M^{-10/39}} \\
&= \frac{8}{3} \frac{(\gamma + 1)^4}{\gamma^3} M^{-10/13} + \mathcal{O}(M^{-40/39}) \\
&= \frac{8}{3} \frac{(\gamma + 1)^4}{\gamma^3} M^{-10/13} (1 + \mathcal{O}(M^{-10/39})) \\
&> \int_0^{M^{10/39}} 3\epsilon'' t^2 dt.
\end{aligned} \tag{5.42}$$

Hence we know that for $w \in \bar{\Sigma}_{\text{remote}}^M$,

$$\Re \left(\frac{(\gamma + 1)^4}{3\gamma^3} M w^3 + M R_1(w) \right) > M \int_0^{M^{10/39}} 3\epsilon'' t^2 dt = M \epsilon'' l^3, \tag{5.43}$$

and have the estimation that if $\xi \geq T$, for $0 < \epsilon''' < \epsilon''$ and M large enough, ($l \geq M^{-10/39}$)

$$\begin{aligned}
& |F_M(\xi, w)| \\
&< \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} e^{-(\xi - T) \frac{(\gamma + 1)^{4/3}}{\gamma}} M^{1/3} l^{-(\epsilon'' l^3 + T \frac{(\gamma + 1)^{4/3}}{\gamma} M^{-2/3} l) M} \\
&\quad \times \left| \frac{\left(w + \frac{\gamma}{\gamma + 1} \right)^{r - r'}}{\left(w - \frac{1}{\gamma + 1} \right)^r} \left(\prod_{j=1}^{s' - 1} \left(w + \frac{a_j - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_j)} \right)^{r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_{s'})} \right)^{t' - 1} \right| \\
&< e^{-(\xi - T) \frac{(\gamma + 1)^{4/3}}{\gamma}} M^{1/13} e^{-\epsilon''' M^{3/13}}.
\end{aligned} \tag{5.44}$$

Now we get the result by similar inequalities as (5.35)–(5.37). \square

If we define

$$\bar{C}_{r'} = \left(\prod_{j=1}^{s' - 1} \left(\frac{a_j - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_j)} \right)^{r_j} \right) \left(\frac{a_{s'} - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_{s'})} \right)^{t'}, \tag{5.45}$$

we have

Lemma 5.3. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_{\text{local}}^M} F_M(\xi, w) dw - (-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1} \text{Ai}(\xi) \right| < \frac{1}{3} \frac{e^{-\xi/2}}{M^{1/40}}, \quad (5.46)$$

for any $\xi \geq T$.

Proof. On $\bar{\Sigma}_{\text{local}}^M$, $|w| < 2M^{-10/39}$, and $R_1(w) = \mathcal{O}(M^{-40/39})$, so that

$$\begin{aligned} F_M(\xi, w) &= \frac{\left(\frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(-\frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(\frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{r_j} \right) \left(\frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})} \right)^{t'-1} \\ &\quad \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} e^{\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi w - \frac{(\gamma+1)^4}{3\gamma^3} M w^3} (1 + \mathcal{O}(M^{-1/39})), \end{aligned} \quad (5.47)$$

and the $\mathcal{O}(M^{-1/39})$ term is independent of ξ . After the substitution $u = \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} w$, we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\bar{\Sigma}_{\text{local}}^M} F_M(\xi, w) dw &= \\ &= \frac{(-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1}}{2\pi i} \int_{\substack{\bar{\Sigma}^\infty \\ > -\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/13}}} e^{\xi u - \frac{u^3}{3}} du (1 + \mathcal{O}(M^{-1/39})). \end{aligned} \quad (5.48)$$

On $\bar{\Sigma}^\infty$, if $\xi \geq T$, $|e^{(\xi-T)u}| \leq e^{T/2}e^{-\xi/2}$, and we have

$$\begin{aligned}
& \left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_{\text{local}}^M} F_M(\xi, w) dw - (-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1} \text{Ai}(\xi) \right| \\
& \leq e^{T/2} e^{-\xi/2} \left| \frac{(-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1}}{2\pi i} \int_{\bar{\Sigma}^\infty} \left| e^{Tu - \frac{u^3}{3}} \right| du (1 + \mathcal{O}(M^{-1/39})) \right| \\
& \quad + e^{T/2} e^{-\xi/2} \left| \frac{(-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1}}{2\pi i} \int_{\bar{\Sigma}^\infty} \left| e^{Tu - \frac{u^3}{3}} \right| du \mathcal{O}(M^{-1/39}) \right|,
\end{aligned} \tag{5.49}$$

and we can get the result by direct calculation. \square

By lemmas 5.1–5.3, and (5.29), we get the convergence result

$$\begin{aligned}
& \left| \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi_{r'}(p+q\xi) - \right. \\
& \quad \left. (-1)^r \gamma^{r-r'} (\gamma+1)^{r'} \bar{C}_{r'-1} \text{Ai}(\xi) \right| < \frac{e^{-\xi/2}}{M^{1/40}}. \tag{5.50}
\end{aligned}$$

In the same way, we have the result for $\psi(p+q\xi)$

$$\left| \frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi(p+q\xi) - (-\gamma)^r \text{Ai}(\xi) \right| < \frac{e^{-\xi/2}}{M^{1/40}}. \tag{5.51}$$

5.1.2 Asymptotics of $\tilde{\psi}(p+q\eta)$ and $\tilde{\psi}_{r'}(p+q\eta)$

We only analyze $\tilde{\psi}_{r'}(p+q\eta)$, The analysis of $\psi(p+q\eta)$ is similar and simpler, and we only give the result.

We have

$$e^{-Myz} \frac{z^M}{(z-1)^N} = e^{Mf(z) - \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta z}, \tag{5.52}$$

where $f(z)$ is defined by (5.17). Now we can write (2.78) as

$$\begin{aligned} \tilde{\psi}_{r'}(y) &= \frac{1}{2\pi i} \\ \oint_{\Gamma} e^{Mf(z) - \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta z} \frac{(z-1)^r}{z^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a_{s'}} \right)^{-t'} dz. \end{aligned} \quad (5.53)$$

After the substitution $z = w + \frac{\gamma}{\gamma+1}$, we get

$$\begin{aligned} &\tilde{\psi}_{r'}(p + q\eta) \\ &= \frac{1}{2\pi i} \oint_{\bar{\Gamma}^M} e^{M - \left(\frac{\gamma+1}{\gamma} + \log \gamma - (1-\gamma^{-2}) \log(\gamma+1) + \gamma^{-2} \pi i + \frac{(\gamma+1)^4}{3\gamma^3} w^3 + R_1(w) \right) - \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta \left(w + \frac{\gamma}{\gamma+1} \right)} \\ &\quad \frac{\left(w - \frac{1}{\gamma+1} \right)^r}{\left(w + \frac{\gamma}{\gamma+1} \right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{-r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})} \right)^{-t'} dw \\ &= \frac{(-1)^N}{2\pi i} \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \oint_{\bar{\Gamma}^M} e^{-\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta w + \frac{(\gamma+1)^4}{3\gamma^3} M w^3 + M R_1(w)} \\ &\quad \frac{\left(w - \frac{1}{\gamma+1} \right)^r}{\left(w + \frac{\gamma}{\gamma+1} \right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{-r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1+\gamma^{-1})(1+a_{s'})} \right)^{-t'} dw, \end{aligned} \quad (5.54)$$

where $\bar{\Gamma}^M$ is a contour containing $\frac{1}{\gamma+1}$ and $\frac{\gamma^{-1}-a_j}{(1+\gamma^{-1})(1+a_j)}$, ($j = 1, \dots, s$), composed of $\bar{\Gamma}_1^M$, $\bar{\Gamma}_2^M$, $\bar{\Gamma}_3^M$, $\bar{\Gamma}_4^M$, $\bar{\Gamma}_5^M$ and $\bar{\Gamma}_6^M$, which are defined as below, with the constant C_{right} a large enough positive number, so that $\bar{\Gamma}^M$ contains all the poles if M is large

enough.

$$\bar{\Gamma}_1^M = \left\{ \left(\frac{2}{\gamma} - t \right) \frac{\gamma}{\gamma+1} e^{\frac{\pi i}{3}} \middle| 0 \leq t \leq \frac{2}{\gamma} - \frac{1}{(\gamma+1)^{1/3}} M^{-1/3} \right\}, \quad (5.55)$$

$$\bar{\Gamma}_2^M = \left\{ \frac{\gamma}{(\gamma+1)^{4/3}} M^{-1/3} e^{-t\pi i} \middle| -\frac{\pi}{3} \leq t \leq \frac{\pi}{3} \right\}, \quad (5.56)$$

$$\bar{\Gamma}_3^M = \left\{ t \frac{\gamma}{\gamma+1} e^{\frac{5\pi i}{3}} \middle| \frac{1}{(\gamma+1)^{1/3}} M^{-1/3} \leq t \leq \frac{2}{\gamma} \right\}, \quad (5.57)$$

$$\bar{\Gamma}_4^M = \left\{ -t + \frac{\sqrt{3}}{\gamma+1} i \middle| -C_{\text{right}} \leq t \leq -\frac{1}{\gamma+1} \right\}, \quad (5.58)$$

$$\bar{\Gamma}_5^M = \left\{ t - \frac{\sqrt{3}}{\gamma+1} i \middle| \frac{1}{\gamma+1} \leq t \leq C_{\text{right}} \right\}, \quad (5.59)$$

$$\bar{\Gamma}_6^M = \left\{ C_{\text{right}} + it \middle| -\frac{\sqrt{3}}{(\gamma+1)} \leq t \leq \frac{\sqrt{3}}{(\gamma+1)} \right\}. \quad (5.60)$$

For asymptotic analysis, we define

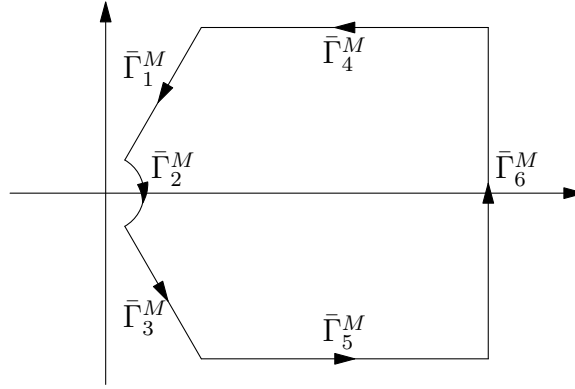


Figure 5.4: $\bar{\Gamma}^M$

$$\bar{\Gamma}_{\text{local}}^M = \{z \in \bar{\Gamma}^M \mid \Re(z) < M^{-10/39}\}, \quad (5.61)$$

$$\bar{\Gamma}_{\text{remote}}^M = (\bar{\Gamma}_1^M \cup \bar{\Gamma}_3^M) \setminus \bar{\Gamma}_{\text{local}}^M. \quad (5.62)$$

If we denote

$$G_M(\eta, w) = \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} e^{-\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta w + \frac{(\gamma+1)^4}{3\gamma^3} M w^3 + M R_1(w)} \\ \frac{\left(w - \frac{1}{\gamma+1}\right)^r}{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_j)} \right)^{-r_j} \right) \left(w + \frac{a_{s'} - \gamma^{-1}}{(1 + \gamma^{-1})(1 + a_{s'})} \right)^{-t'}, \quad (5.63)$$

we have

$$(-1)^N \frac{(\gamma + 1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} M y} \tilde{\psi}_{r'}(p + q\eta) = \frac{1}{2\pi i} \oint_{\bar{\Gamma}^M} G_M(\eta, w) dw, \quad (5.64)$$

and establish several lemmas:

Lemma 5.4. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}_4^M \cup \bar{\Gamma}_5^M \cup \bar{\Gamma}_6^M} G_M(\eta, w) dw \right| < \frac{1}{3} \frac{e^{-\eta/2}}{M^{1/40}}, \quad (5.65)$$

for any $\eta \geq T$.

Lemma 5.5. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}_{\text{remote}}^M} G_M(\eta, w) dw \right| < \frac{1}{3} \frac{e^{-\eta/2}}{M^{1/40}}, \quad (5.66)$$

for any $\eta \geq T$.

Lemma 5.6. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}_{\text{local}}^M} F_M(\eta, w) dw - \frac{(-1)^{r-1} (1 + \gamma^{-1})}{\gamma^{r-r'} (\gamma + 1)^{r'} \bar{C}_{r'}} \text{Ai}(\eta) \right| < \frac{1}{3} \frac{e^{-\eta/2}}{M^{1/40}}, \quad (5.67)$$

for any $\eta \geq T$.

Since the proofs of lemma 5.4–5.6 are similar to those of lemmas 5.1–5.3, we only give an outline of the proof of lemma 5.6.

Sketch of proof of lemma 5.6. On $\bar{\Gamma}_{\text{local}}^M$, $|w| < 2M^{-10/39}$ and $R_1(w) = \mathcal{O}(M^{-40/39})$, so that

$$G_M(\eta, w) = \frac{(-1)^r(1 + \gamma^{-1})}{\gamma^{r-r'}(\gamma + 1)^{r'}\bar{C}_{r'}} \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} e^{-\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta w + \frac{(\gamma+1)^4}{3\gamma^3} M w^3} (1 + \mathcal{O}(M^{-1/39})), \quad (5.68)$$

after the substitution $u = \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} w$, we get

$$\int_{\bar{\Gamma}_{\text{local}}^M} G_M(\eta, w) dw = \frac{(-1)^r(1 + \gamma^{-1})}{\gamma^{r-r'}(\gamma + 1)^{r'}\bar{C}_{r'}} \int_{\bar{\Gamma}^\infty} e^{-\eta u + \frac{u^3}{3}} du (1 + \mathcal{O}(M^{-1/39})). \quad (5.69)$$

Also we have that on $\bar{\Gamma}^\infty$, if $\eta \geq T$, $|e^{-(\eta-T)u}| \leq e^{T/2} e^{-\eta/2}$. We can prove lemma 5.6 in the same way as proving lemma 5.3. \square

By lemmas 5.4–5.6, and (5.64), we get the convergence result

$$\left| \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{(\gamma + 1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} M y} \tilde{\psi}_{r'}(p + q\eta) - \frac{(-1)^{r-1}(1 + \gamma^{-1})}{\gamma^{r-r'}(\gamma + 1)^{r'}\bar{C}_{r'}} \text{Ai}(\xi) \right| < \frac{e^{-\eta/2}}{M^{1/40}}. \quad (5.70)$$

And in the same way, we have the result for $\tilde{\psi}(p + q\xi)$

$$\left| \frac{(\gamma + 1)^{4/3}}{\gamma} M^{1/3} (-1)^N \frac{(\gamma + 1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} M y} \tilde{\psi}(p + q\eta) - (-1)^{r-1} \gamma^{-r} \text{Ai}(\xi) \right| < \frac{e^{-\eta/2}}{M^{1/40}}. \quad (5.71)$$

5.2 Asymptotics of $\psi_{r'}(p + q\xi)$ and $\tilde{\psi}_{r'}(p + q\eta)$ when

$$a_{s'} = \gamma^{-1}$$

In this section, we still assume $x = p + q\xi$, $y = p + q\eta$, $p = (1 + \gamma^{-1})^2$ and $q = \frac{(1+\gamma)^{4/3}}{\gamma M^{2/3}}$.

With $\bar{\Sigma}^\infty$ defined in last section, we have

$$\frac{1}{2\pi i} \int_{\bar{\Sigma}^\infty} e^{\xi u - \frac{u^3}{3}} u^{t'-1} du = (-1)^{t'-1} t^{(t')}(\xi), \quad (5.72)$$

which can be proved by a simple change of variable similar to (5.7). For any T , if $-c$ is large enough

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}_{\leq c}^\infty} e^{Tu - \frac{u^3}{3}} u^{t'-1} du \right| < \frac{1}{-c}. \quad (5.73)$$

we define $\bar{\Gamma}^\infty = \bar{\Gamma}_1^\infty \cup \bar{\Gamma}_2^\infty \cup \bar{\Gamma}_3^\infty$, where

$$\bar{\Gamma}_1^\infty = \{-te^{\frac{\pi i}{3}} \mid t \leq -\frac{1}{6}\}, \quad (5.74)$$

$$\bar{\Gamma}_2^\infty = \{\frac{1}{6}e^{t\pi i} \mid \frac{1}{3} \leq t \leq \frac{5}{3}\}, \quad (5.75)$$

$$\bar{\Gamma}_3^\infty = \{te^{\frac{5\pi i}{3}} \mid t \geq \frac{1}{6}\}, \quad (5.76)$$

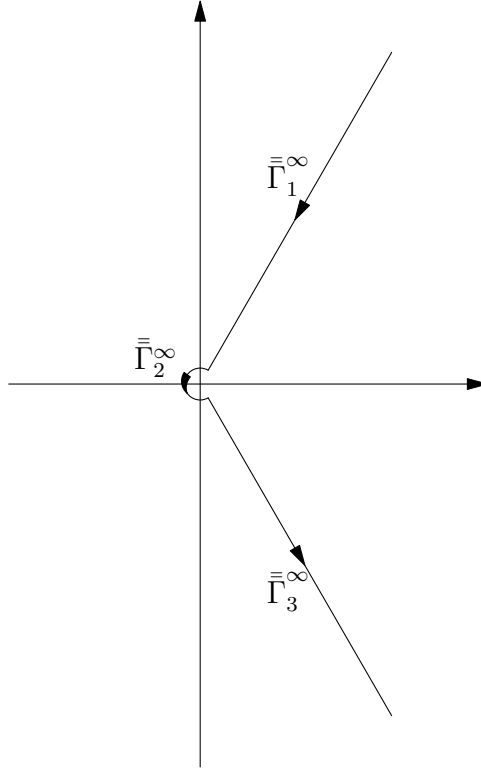
and

$$\bar{\Gamma}_{<c}^\infty = \{w \in \bar{\Gamma}^\infty \mid \Re(w) < c\}, \quad (5.77)$$

$$\bar{\Gamma}_{\geq c}^\infty = \bar{\Gamma}^\infty \setminus \bar{\Sigma}_{<c}^\infty. \quad (5.78)$$

We have

$$\frac{1}{2\pi i} \int_{\bar{\Gamma}^\infty} e^{-\eta u + \frac{u^3}{3}} \frac{du}{u^{t'}} = (-1)^{t'-1} s^{(t')}(\xi), \quad (5.79)$$


 Figure 5.5: $\bar{\Gamma}^\infty$

and for any T , if c is large enough

$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}_{\geq c}^\infty} e^{-Tu + \frac{u^3}{3}} \frac{du}{u^{t'}} \right| < \frac{1}{c}. \quad (5.80)$$

5.2.1 Asymptotics of $\psi_{r'}(p + q\xi)$

Similar to (5.18), we have ($f(z)$ is defined by (5.17))

$$\begin{aligned} \psi_{r'}(x) &= \frac{1}{2\pi i} \\ \oint_{\Sigma} e^{-Mf(z) + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi z} \frac{z^{r-r'}}{(z-1)^r} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{\gamma}{\gamma+1} \right)^{t'-1} dz, \end{aligned} \quad (5.81)$$

and after the substitution $z = w + \frac{\gamma}{\gamma+1}$, we get

$$\begin{aligned}
& \psi_{r'}(p + q\xi) \\
&= \frac{1}{2\pi i} \oint_{\bar{\Sigma}^M} e^{M\left(\frac{\gamma+1}{\gamma} - \log \gamma + (1-\gamma^{-2}) \log(\gamma+1) - \gamma^{-2} \pi i - \frac{(\gamma+1)^4}{3\gamma^3} w^3 - R_1(w)\right) + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi \left(w + \frac{\gamma}{\gamma+1}\right)} \\
& \quad \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{r_j} \right) w^{t'-1} dw \\
&= \frac{(-1)^N}{2\pi i} \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1} Mx} \oint_{\bar{\Sigma}^M} e^{\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi w - \frac{(\gamma+1)^4}{3\gamma^3} M w^3 - M R_1(w)} \\
& \quad \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{r_j} \right) w^{t'-1} dw,
\end{aligned} \tag{5.82}$$

with $\bar{\Sigma}^M$ defined by (5.22)–(5.25).

Now we denote

$$\begin{aligned}
F_{Mt'}(\xi, w) &= \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{t'} e^{\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \xi w - \frac{(\gamma+1)^4}{3\gamma^3} M w^3 - M R_1(w)} \\
& \quad \frac{\left(w + \frac{\gamma}{\gamma+1}\right)^{r-r'}}{\left(w - \frac{1}{\gamma+1}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{r_j} \right) w^{t'-1}, \tag{5.83}
\end{aligned}$$

and have

$$(-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \psi_{r'}(p + q\xi) = \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{-t'} \frac{1}{2\pi i} \oint_{\bar{\Sigma}^M} F_{Mt'}(\xi, w) dw, \tag{5.84}$$

Similar to the $a_{s'} < \gamma^{-1}$ case, we have

Lemma 5.7. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Sigma}^M} F_{Mt'}(\xi, w) dw - (-1)^r \gamma^{r-r'} (\gamma+1)^r \bar{C}_{r'-t'} (-1)^{t'-1} t^{(t)}(\xi) \right| < \frac{e^{-\xi/2}}{M^{1/40}}, \tag{5.85}$$

for any $\xi \geq T$.

Proof. Similar to the proofs of lemmas 5.1–5.3 together. \square

Hence we have the convergence result

$$\left| \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{t'} (-1)^N \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} e^{\xi/3} \psi_{r'}(p+q\xi) \right. \\ \left. - (-1)^r \gamma^{r-r'} (\gamma+1)^r \bar{C}_{r'-t'} (-1)^{t'-1} e^{\xi/3} t^{(t')}(\xi) \right| < \frac{e^{-\xi/6}}{M^{1/40}}. \quad (5.86)$$

5.2.2 Asymptotics of $\tilde{\psi}_{r'}(p+q\eta)$

Similar to (5.53), we have

$$\tilde{\psi}_{r'}(y) = \frac{1}{2\pi i} \oint_{\Gamma} e^{Mf(z) - \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta z} \frac{(z-1)^r}{z^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{\gamma}{\gamma+1} \right)^{-t'} dz, \quad (5.87)$$

and after the substitution $z = w + \frac{\gamma}{\gamma+1}$, we get

$$\begin{aligned} & \tilde{\psi}_{r'}(p+q\eta) \\ &= \frac{1}{2\pi i} \oint_{\bar{\Gamma}_M} e^{M \left(\frac{\gamma+1}{\gamma} + \log \gamma - (1-\gamma^{-2}) \log(\gamma+1) + \gamma^{-2} \pi i + \frac{(\gamma+1)^4}{3\gamma^3} w^3 + R_1(w) \right) - \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta \left(w + \frac{\gamma}{\gamma+1} \right)} \\ & \quad \frac{\left(w - \frac{1}{\gamma+1} \right)^r}{\left(w + \frac{\gamma}{\gamma+1} \right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{-r_j} \right) w^{-t'} dw \\ &= \frac{(-1)^N}{2\pi i} \frac{\gamma^M}{(\gamma+1)^{M-N}} e^{-\frac{\gamma}{\gamma+1} Mx} \oint_{\bar{\Gamma}_M} e^{-\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta w + \frac{(\gamma+1)^4}{3\gamma^3} Mw^3 + MR_1(w)} \\ & \quad \frac{\left(w - \frac{1}{\gamma+1} \right)^r}{\left(w + \frac{\gamma}{\gamma+1} \right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{-r_j} \right) w^{-t'} dw, \end{aligned} \quad (5.88)$$

where $\bar{\bar{\Gamma}}^M$ is slightly different from $\bar{\Gamma}^M$: $\bar{\bar{\Gamma}}^M$ is composed of $\bar{\bar{\Gamma}}_1^M, \dots, \bar{\bar{\Gamma}}_6^M$, which are

$$\bar{\bar{\Gamma}}_1^M = \left\{ \left(\frac{2}{\gamma} - t \right) \frac{\gamma}{\gamma+1} e^{\frac{\pi i}{3}} \middle| 0 \leq t \leq \frac{2}{\gamma} - \frac{1/6}{(\gamma+1)^{1/3}} M^{-1/3} \right\}, \quad (5.89)$$

$$\bar{\bar{\Gamma}}_2^M = \left\{ \frac{\gamma/6}{(\gamma+1)^{4/3}} M^{-1/3} e^{t\pi i} \middle| \frac{1}{3} \leq t \leq \frac{5}{3} \right\}, \quad (5.90)$$

$$\bar{\bar{\Gamma}}_3^M = \left\{ t \frac{\gamma}{\gamma+1} e^{\frac{5\pi i}{3}} \middle| \frac{1/6}{(\gamma+1)^{1/3}} M^{-1/3} \leq t \leq \frac{2}{\gamma} \right\}, \quad (5.91)$$

$$\bar{\bar{\Gamma}}_*^M = \bar{\Gamma}_*^M, \text{ for } * = 4, 5, 6. \quad (5.92)$$

We also define

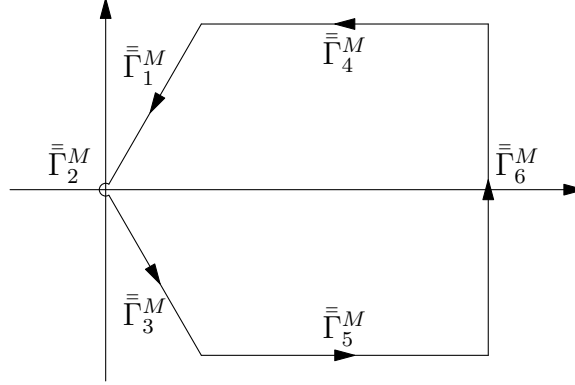


Figure 5.6: $\bar{\bar{\Gamma}}^M$

$$\bar{\bar{\Gamma}}_{\text{local}}^M = \{z \in \bar{\bar{\Gamma}}^M \mid \Re(z) < M^{-10/39}\}, \quad (5.93)$$

$$\bar{\bar{\Gamma}}_{\text{remote}}^M = (\bar{\bar{\Gamma}}_1^M \cup \bar{\bar{\Gamma}}_3^M) \setminus \bar{\bar{\Gamma}}_{\text{local}}^M. \quad (5.94)$$

If we denote

$$G_{Mt'}(\eta, w) = \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{1-t'} e^{-\frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} \eta w + \frac{(\gamma+1)^4}{3\gamma^3} M w^3 + M R_1(w)} \frac{\left(w - \frac{1}{\gamma+1} \right)^r}{\left(w + \frac{\gamma}{\gamma+1} \right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - \gamma^{-1}}{(1+\gamma^{-1})(1+a_j)} \right)^{-r_j} \right) w^{-t'}, \quad (5.95)$$

we have

$$(-1)^N \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1}My} \tilde{\psi}_{r'}(p+q\eta) = \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{t'-1} \frac{1}{2\pi i} \oint_{\bar{\Gamma}^M} G_{Mt'}(\eta, w) dw, \quad (5.96)$$

and

Lemma 5.8. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\bar{\Gamma}^M} G_{Mt'}(\eta, w) dw - \frac{(-1)^r(1+\gamma^{-1})}{\gamma^{r-r'}(\gamma+1)^{r'}\bar{C}_{r'-t'}} (-1)^{t'-1} s^{(t)}(\eta) \right| < \frac{e^{\eta/6}}{M^{1/40}}, \quad (5.97)$$

for any $\eta \geq T$.

Sketch of proof. The integral of $G_{Mt'}(\eta, w)$ over $\bar{\Gamma}^M \setminus \bar{\Gamma}_{\text{local}}^M$ is negligible, and we can estimate it as lemmas 5.4 and 5.5, and get the same result. On $\bar{\Gamma}_{\text{local}}^M$, if $\eta \geq T$, $|e^{-(\eta-T)u}| \leq e^{-T/6}e^{\eta/6}$, and we can carry out the proof like that of lemma 5.6, with the upper bound of $|e^{-(\eta-T)u}|$ changed from $e^{T/2}e^{-\eta/2}$ to $e^{-T/6}e^{\eta/6}$. \square

Therefore we have the convergence result

$$\left| \left(\frac{(\gamma+1)^{4/3}}{\gamma} M^{1/3} \right)^{1-t'} (-1)^N \frac{(\gamma+1)^{M-N}}{\gamma^M} e^{\frac{\gamma}{\gamma+1}My} e^{-\eta/3} \tilde{\psi}_{r'}(p+q\eta) - \frac{(-1)^r(1+\gamma^{-1})}{\gamma^{r-r'}(\gamma+1)^{r'}\bar{C}_{r'-t'}} (-1)^{t'-1} e^{-\eta/3} s^{(t')}(\eta) \right| < \frac{e^{-\eta/6}}{M^{1/40}}. \quad (5.98)$$

5.3 Asymptotics of $\psi(p+q\xi)$, $\psi_{r'}(p+q\xi)$, $\tilde{\psi}(p+q\eta)$

and $\tilde{\psi}_{r'}(p+q\eta)$ when $a_{s'} \leq a$

In this section, a is a member greater than γ^{-1} , and we assume $p = (1+a) \left(1 + \frac{1}{\gamma^2 a}\right)$ and $q = (1+a) \sqrt{1 - \frac{1}{\gamma^2 a^2}} \frac{1}{\sqrt{M}}$.

We define

$$\tilde{\Sigma}^\infty = \{it - 1 \mid -\infty \leq t \leq \infty\}, \quad (5.99)$$

$$\tilde{\Sigma}_{<c}^\infty = \{w \in \tilde{\Sigma}^\infty \mid |w| \leq c\}, \quad (5.100)$$

$$\tilde{\Sigma}_{\geq c}^\infty = \tilde{\Sigma}^\infty \setminus \tilde{\Sigma}_{<c}^\infty, \quad (5.101)$$

and for any T , if c is large enough,

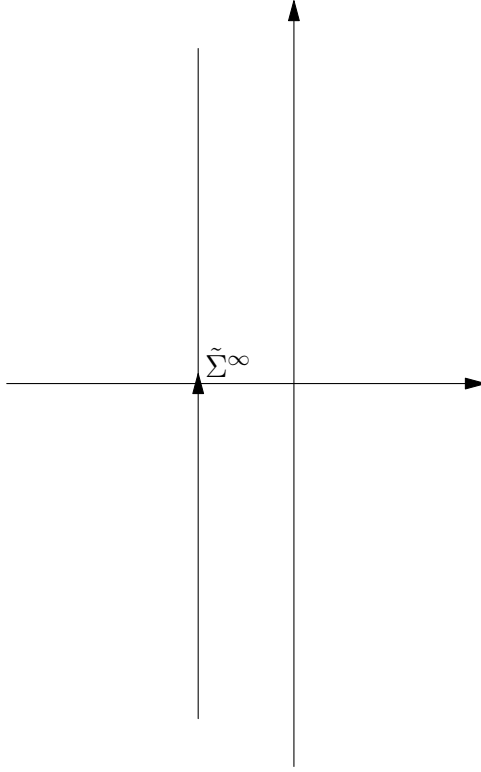


Figure 5.7: $\tilde{\Sigma}^\infty$

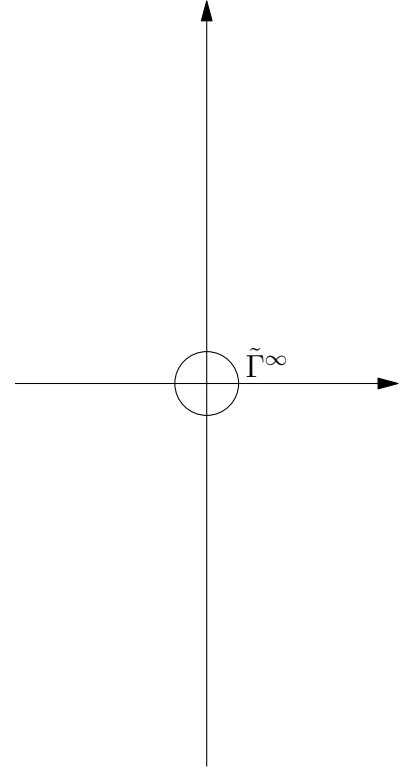


Figure 5.8: $\tilde{\Gamma}^\infty$

$$\left| \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\geq c}^\infty} e^{Tu + \frac{u^2}{2}} u^{t'-1} du \right| < \frac{1}{c}. \quad (5.102)$$

We also define

$$\tilde{\Gamma}^\infty = \{e^{it}/3 \mid 0 \leq t \leq 2\pi\}. \quad (5.103)$$

We need two integral representations of Hermite polynomials. First, we have the

explicit formulas for $H_n(x)$

$$H_{2k}(x) = (-1)^k (2k-1)!! \left(1 + \sum_{l=1}^k \frac{(-2k)(-2k+2)\dots(-2k+2j-2)}{(2j)!} x^{2j} \right), \quad (5.104)$$

$$H_{2k+1}(x) = (-1)^k (2k+1)!! \left(x + \sum_{l=1}^k \frac{(-2k)(-2k+2)\dots(-2k+2j-2)}{(2j)!} x^{2j+1} \right). \quad (5.105)$$

On the other hand, ($v = u + \xi$)

$$\begin{aligned} \frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\xi u + \frac{u^2}{2}} u^{t'-1} du &= e^{\frac{\xi^2}{2}} \frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\frac{(\xi+u)^2}{2}} u^{t'-1} du \\ &= e^{-\frac{\xi^2}{2}} \frac{1}{2\pi i} \int_{-\infty i + \xi}^{\infty i + \xi} e^{\frac{v^2}{2}} (v - \xi)^{t'-1} dv \end{aligned} \quad (5.106)$$

$$= e^{-\frac{\xi^2}{2}} \frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\frac{v^2}{2}} (v - \xi)^{t'-1} dv, \quad (5.107)$$

where the integral in (5.106) is along the vertical line parallel to $\tilde{\Sigma}^\infty$ through the point $v = \xi$, and the equivalency of (5.106) and (5.107) is a simple application of the Cauchy integral formula. Now if we write

$$\frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\frac{v^2}{2}} (v - \xi)^{t'-1} dv = \sum_{j=0}^{t'-1} \binom{t'-1}{j} (-\xi)^{t'-j-1} \frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\frac{v^2}{2}} v^j dv, \quad (5.108)$$

and

$$\frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\frac{v^2}{2}} v^j dv = \frac{i^j}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x^j dx = \begin{cases} 0 & j \text{ odd,} \\ \frac{(-1)^k}{\sqrt{2\pi}} (2k-1)!! & j = 2k. \end{cases} \quad (5.109)$$

Compare (5.107), (5.108) and (5.109) to (5.104) and (5.105), we get

$$\frac{1}{2\pi i} \int_{\tilde{\Sigma}^\infty} e^{\xi u + \frac{u^2}{2}} u^{t'-1} du = (-1)^{t'-1} \frac{H_{t'-1}(\xi)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}. \quad (5.110)$$

The second integral representation of Hermite polynomials is more familiar:

$$\frac{1}{2\pi i} \int_{\tilde{\Gamma}^\infty} e^{-\eta u - \frac{u^2}{2}} \frac{du}{u^{t'}} = \frac{(-1)^{t'-1}}{(t'-1)!} H_{t'-1}(\eta). \quad (5.111)$$

5.3.1 Asymptotics of $\psi(p + q\xi)$ and $\psi_{r'}(p + q\xi)$

We only consider $\psi_{r'}$ with $a_{s'} = a$. $\psi_{r'}$ with $a_{s'} < a$ and ψ can be solved similarly, and we only give the results.

We have

$$e^{Mxz} \frac{(z-1)^N}{z^M} = e^{-Mg(z) + (1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\xi z}, \quad (5.112)$$

where

$$g(z) = -(a+1) \left(1 + \frac{1}{\gamma^2 a}\right) z + \log z - \gamma^{-2} \log(z-1). \quad (5.113)$$

Now we can write (2.77) as

$$\begin{aligned} \psi_{r'}(x) = \frac{1}{2\pi i} \oint_{\Sigma} e^{-Mg(z) + (1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\xi z} \\ \frac{z^{r-r'}}{(z-1)^r} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{r_j} \right) \left(z - \frac{1}{1+a} \right)^{t'-1} dz. \end{aligned} \quad (5.114)$$

For $g(z)$, we have

- $g'(z) = -(a+1) \left(1 + \frac{1}{\gamma^2 a}\right) + \frac{1}{z} - \frac{\gamma^{-2}}{z-1}$, with zero points $z = \frac{\gamma^2 a}{1 + \gamma^2 a}$ and $z = \frac{1}{1+a}$.
- $g''(z) = -\frac{1}{z^2} + \frac{\gamma^{-2}}{(z-1)^2}$, $g''\left(\frac{\gamma^2 a}{1 + \gamma^2 a}\right) = (\gamma^{-1} + \gamma a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) > 0$ and $g''\left(\frac{1}{1+a}\right) = -(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) < 0$.

Hence locally around $z = \frac{1}{1+a}$,

$$g\left(\frac{1}{1+a} + w\right) = -\left(1 + \frac{1}{\gamma^2 a}\right) - (1 - \gamma^{-2}) \log(1+a) - \gamma^{-2} \log a + \gamma^{-2} \pi i \\ - \frac{1}{2}(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) w^2 + R_2(w), \quad (5.115)$$

where

$$R_2(w) = \mathcal{O}(w^3), \text{ as } w \rightarrow \infty. \quad (5.116)$$

After the substitution $z = w + \frac{1}{1+a}$, we get by (5.114)

$$\begin{aligned} & \psi_{r'}(p + q\xi) \\ &= \frac{1}{2\pi i} \oint_{\tilde{\Sigma}^M} e^{M\left(\left(1+\frac{1}{\gamma^2 a}\right) + (1-\gamma^{-2}) \log(1+a) + \gamma^{-2} \log a - \gamma^{-2} \pi i + \frac{1}{2}(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) w^2 - R_2(w)\right)} \\ & \quad e^{(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)M\xi\left(w+\frac{1}{1+a}\right)}} \frac{\left(w + \frac{1}{1+a}\right)^{r-r'}}{\left(w - \frac{a}{1+a}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)}\right)^{r_j}\right) w^{t'-1} dw \\ &= \frac{(-1)^N}{2\pi i} a^N (1+a)^{M-N} e^{\frac{M}{1+a}x} \oint_{\tilde{\Sigma}^M} e^{\frac{1}{2}(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) M w^2 - M R_2(w)} \\ & \quad \frac{\left(w + \frac{1}{1+a}\right)^{r-r'}}{\left(w - \frac{a}{1+a}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)}\right)^{r_j}\right) w^{t'-1} dw, \end{aligned} \quad (5.117)$$

where $\tilde{\Sigma}^M$ is a contour around $w = -\frac{1}{1+a}$, composed of $\tilde{\Sigma}_1^M$, $\tilde{\Sigma}_2^M$, $\tilde{\Sigma}_3^M$ and $\tilde{\Sigma}_4^M$, which are defined as ($q = (1+a)\sqrt{1 - \frac{1}{\gamma^2 a^2} \frac{1}{\sqrt{M}}}$)

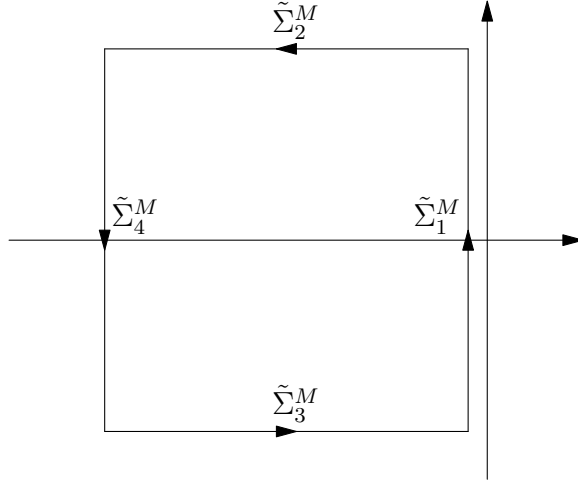
$$\tilde{\Sigma}_1^M = \{it - q^{-1}/M \mid -2 \leq t \leq 2\}, \quad (5.118)$$

$$\tilde{\Sigma}_2^M = \{2i - t \mid q^{-1}/M \leq t \leq 4\}, \quad (5.119)$$

$$\tilde{\Sigma}_3^M = \{-4 - it \mid -2 \leq t \leq 2\}, \quad (5.120)$$

$$\tilde{\Sigma}_4^M = \{t - 2i \mid -4 \leq t \leq -q^{-1}/M\}, \quad (5.121)$$

and for the asymptotic analysis, we define

Figure 5.9: $\tilde{\Sigma}^M$

$$\tilde{\Sigma}_{\text{local}}^M = \{w \in \tilde{\Sigma}^M \mid |w| \leq M^{-2/5}\}, \quad (5.122)$$

$$\tilde{\Sigma}_{\text{remote}}^M = \tilde{\Sigma}_1^M \setminus \tilde{\Sigma}_{\text{local}}^M. \quad (5.123)$$

Now if we denote

$$F_{Mat'}(w) = \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{t'} e^{(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \xi w + \frac{1}{2}(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) M w^2 - M R_2(w)} \frac{\left(w + \frac{1}{1+a}\right)^{r-r'}}{\left(w - \frac{a}{1+a}\right)^r} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)} \right)^{r_j} \right) w^{t'-1}, \quad (5.124)$$

We have

$$\left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{t'} \frac{e^{-\frac{M}{1+a} x}}{(-a)^N (1+a)^{M-N}} \psi_{r'}(p+q\xi) = \frac{1}{2\pi i} \oint_{\tilde{\Sigma}^M} F_{Mat'}(\xi, w) dw, \quad (5.125)$$

and results similar to lemmas 5.1–5.3:

Lemma 5.9.

$$\left| \frac{1}{2\pi i} \int_{\tilde{\Sigma}_2^M \cup \tilde{\Sigma}_3^M \cup \tilde{\Sigma}_4^M} F_{Mat'}(\xi, w) dw \right| < \frac{1}{3} \frac{e^{-\xi}}{M^{1/10}}, \quad (5.126)$$

for any $\xi \geq T$.

Lemma 5.10.

$$\left| \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\text{remote}}^M} F_{Mat'}(\xi, w) dw \right| < \frac{1}{3} \frac{e^{-\xi}}{M^{1/10}}, \quad (5.127)$$

for any $\xi \geq T$.

Lemma 5.11. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\text{local}}^M} F_{Mat'}(\xi, w) dw - \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a,r'-t'}(-1)^{t'-1} \frac{H_{t'-1}(x)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right| < \frac{1}{3} \frac{e^{-\xi}}{M^{1/10}}, \quad (5.128)$$

for any $\xi \geq T$.

Since their proofs are similar to those of lemmas 5.1–5.3, we only give the proof of lemma 5.11.

Proof of lemma 5.11. On $\tilde{\Sigma}_{\text{local}}^M$, $|w| < M^{-2/5}$, and $R_2(w) = \mathcal{O}(M^{-6/5})$, so that

$$\begin{aligned} F_{Mat'}(\xi, w) = & \frac{(1+a)^{r'}}{(-a)^r} \left(\prod_{j=1}^{s'-1} \left(\frac{a_j - a}{(1+a)(1+a_l)} \right)^{r_j} \right) \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2} \right) M} \right)^{t'} \\ & e^{(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2} \right) M} \xi w + \frac{1}{2} (1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2} \right) M w^2} w^{t'-1} (1 + \mathcal{O}(M^{-1/5})), \end{aligned} \quad (5.129)$$

and the $\mathcal{O}(M^{-1/5})$ term is independent of ξ . After the substitution $u = (1 +$

a) $\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) Mw}$, we get

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\text{local}}^M} F_{Mat'}(\xi, w) dw = \\ & \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a, r'-t'} \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\infty}^{<(1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}} M^{1/10}}} e^{\xi u + \frac{u^2}{2}} u^{t'-1} du \left(1 + \mathcal{O}(M^{-1/5})\right). \end{aligned} \quad (5.130)$$

On $\tilde{\Sigma}^M$, if $\xi \geq T$, $|e^{(\xi-T)u}| \leq e^T e^{-\xi}$, and we have

$$\begin{aligned} & \left| \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\text{local}}^M} F_M(\xi, w) dw \chi(\xi) - \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a, r'-t'} (-1)^{t'-1} \frac{H_{t'-1}(x)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right| \\ & \leq e^T e^{-\xi} \left| \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a, r'-t'} \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\infty}^{\geq(1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}} M^{1/10}}} \left| e^{Tu - \frac{u^2}{2}} u^{t'-1} \right| du (1 + \mathcal{O}(M^{-1/5})) \right| \\ & \quad + e^T e^{-\xi} \left| \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a, r'-t'} \frac{1}{2\pi i} \int_{\tilde{\Sigma}_{\infty}^{<(1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}} M^{1/10}}} \left| e^{Tu - \frac{u^2}{2}} u^{t'-1} \right| du \mathcal{O}(M^{-1/5}) \right|, \end{aligned} \quad (5.131)$$

and we can get the result by direct calculation. \square

By lemmas 5.9–5.11 and (5.125), we have the convergence result for $\psi_{r'}$ with $a_{s'} = a$

$$\begin{aligned} & \left| \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{t'} \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}} e^{2\xi/3} \psi_{r'}(p + q\xi) \right. \\ & \quad \left. - \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a, r'-t'} (-1)^{t'-1} e^{2\xi/3} \frac{H_{t'-1}(x)}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right| < \frac{e^{-\xi/3}}{M^{1/10}}. \end{aligned} \quad (5.132)$$

By the same method, we can get the convergence result for $\psi_{r'}(\xi)$ with $r' =$

$1, \dots, r-t$ and $\xi \geq T$

$$\left| (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}} e^{2\xi/3} \psi_{r'}(p+q\xi) - \frac{(1+a)^{r'}}{(-a)^r} \bar{C}_{a,r'-1} \frac{e^{\frac{2\xi}{3} - \frac{\xi^2}{2}}}{\sqrt{2\pi}} \right| < \frac{e^{-\xi/3}}{M^{1/10}}, \quad (5.133)$$

and for $\psi(\xi)$ with $\xi \geq T$

$$\left| (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M \frac{e^{-\frac{M}{1+a}x}}{(-a)^N (1+a)^{M-N}} e^{2\xi/3} \psi(p+q\xi) - (-a)^{-r} \frac{e^{\frac{2\xi}{3} - \frac{\xi^2}{2}}}{\sqrt{2\pi}} \right| < \frac{e^{-\xi/3}}{M^{1/10}}. \quad (5.134)$$

5.3.2 Asymptotics of $\tilde{\psi}(p+q\eta)$ and $\tilde{\psi}_{r'}(p+q\eta)$

We only consider $\tilde{\psi}_{r'}$ with $a_{s'} = a$, and only state the results for $\tilde{\psi}_{r'}$ with $a_{s'} < a$ and $\tilde{\psi}$, since they are simpler.

$\tilde{\psi}_{r'}(p+q\eta)$ is defined in (2.78) by a contour integral with poles $z = 1$ and $\frac{1}{1+a_j}$, $j = 1, \dots, s$ inside the contour. By the residue theorem, the value of $\tilde{\psi}_{r'}(p+q\eta)$ is the sum of residues at these poles. We will see that the contribution of poles other than $\frac{1}{1+a}$ are negligible, and we are going to calculate the residue at $\frac{1}{1+a}$. To consider these two kinds of poles separately, we deform Γ into the sum of two disconnected contours $\Gamma_{\frac{1}{1+a}}$ and Γ_{right} , where $\Gamma_{\frac{1}{1+a}}$ includes $\frac{1}{1+a}$ and excludes other poles, and vice versa for Γ_{right} .

We have

$$e^{-Myz} \frac{(z-1)^N}{z^M} = e^{-Mg(z)+(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)}M\eta z}, \quad (5.135)$$

where $g(z)$ is defined in (5.113). Then we can write (2.78) as

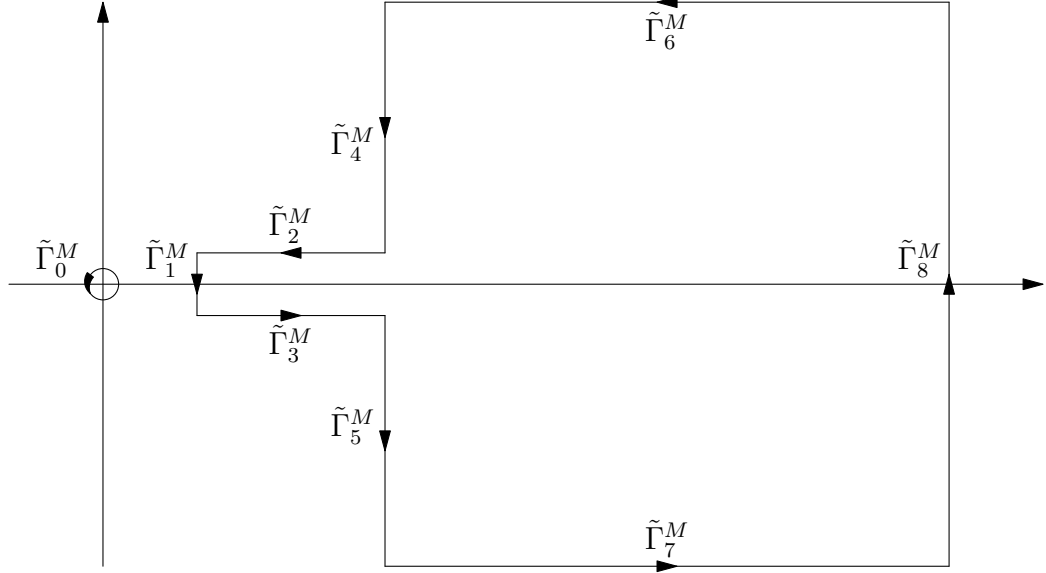
$$\begin{aligned} \tilde{\psi}_{r'}(y) &= \frac{1}{2\pi i} \oint_{\Gamma_{\frac{1}{1+a}} \cup \Gamma_{\text{right}}} e^{-Mg(z) + (1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\eta z} \\ &\quad \frac{(z-1)^r}{z^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a} \right)^{-t'} dz, \end{aligned} \quad (5.136)$$

and after the substitution $z = w + \frac{1}{1+a}$, we get

$$\begin{aligned} &\oint_{\Gamma_{\frac{1}{1+a}}} e^{Mg(z) - (1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\eta z} \frac{(z-1)^r}{z^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(z - \frac{1}{1+a_j} \right)^{-r_j} \right) \left(z - \frac{1}{1+a} \right)^{-t'} dz \\ &= \oint_{\tilde{\Gamma}_0^M} e^{M\left(-\left(1 - \frac{1}{\gamma^2 a^2}\right) - (1-\gamma^{-2})\log(1+a) - \gamma^{-2}\log a + \gamma^{-2}\pi i - \frac{1}{2}(1+a)^2\left(1 - \frac{1}{\gamma^2 a^2}\right)w^2 + R_2(w)\right)} \\ &\quad e^{-(1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\eta\left(w + \frac{1}{1+a}\right)} \frac{\left(w - \frac{a}{1+a}\right)^r}{\left(w + \frac{1}{1+a}\right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)} \right)^{-r_j} \right) \frac{dw}{w^{t'}} \\ &= \frac{(-1)^N}{a^N(1+a)^{M-N}} e^{-\frac{M}{1+a}y} \oint_{\tilde{\Gamma}_0^M} e^{-(1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M\eta w - \frac{1}{2}(1+a)^2\left(1 - \frac{1}{\gamma^2 a^2}\right) M w^2 - M R_2(w)} \\ &\quad \frac{\left(w - \frac{a}{1+a}\right)^r}{\left(w + \frac{1}{1+a}\right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)} \right)^{-r_j} \right) \frac{dw}{w^{t'}}, \end{aligned} \quad (5.137)$$

where $\tilde{\Gamma}_0^M$ is a contour around $w = 0$, defined as

$$\tilde{\Gamma}_0^M = \left\{ \frac{e^{it}/3}{(1+a)\sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right)} M} \middle| 0 \leq t \leq 2\pi \right\}. \quad (5.138)$$


 Figure 5.10: $\tilde{\Gamma}_0^M$ and $\tilde{\Gamma}_{\text{right}}^M$

If we denote

$$\begin{aligned}
 G_{Mat'}(\eta, w) = & \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{1-t'} e^{-(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \eta w - \frac{1}{2}(1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2}\right) M w^2 + M R_2(w)} \\
 & \frac{\left(w - \frac{a}{1+a}\right)^r}{\left(w + \frac{1}{1+a}\right)^{r-r'+1}} \left(\prod_{j=1}^{s'-1} \left(w + \frac{a_j - a}{(1+a)(1+a_j)} \right)^{-r_j} \right) \frac{1}{w^{t'}}, \quad (5.139)
 \end{aligned}$$

We have

$$\begin{aligned}
 & \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{1-t'} (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a} y} \tilde{\psi}_{r'}(p + q\eta) = \\
 & \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_0^M} G_{Mat'}(\eta, w) dw + \oint_{\tilde{\Gamma}_{\text{right}}^M} G_{Mat'}(\eta, w) dw, \quad (5.140)
 \end{aligned}$$

with the definition of $\tilde{\Gamma}_{\text{right}}^M$ to be introduced later, and

Lemma 5.12. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_0^M} G_{Mat'}(\eta, w) dw - \frac{(-a)^r}{(1+a)^{r'-1} \bar{C}_{r'-t'}} \frac{(-1)^{t'-1}}{(t'-1)!} H_{t'-1}(\eta) \right| < \frac{1}{2} \frac{e^{\eta/3}}{M^{1/3}}, \quad (5.141)$$

for any $\eta \geq T$.

Proof. On $\tilde{\Gamma}_0^M$, $|w| = M^{-1/2}$, and $R_2(w) = \mathcal{O}(M^{-3/2})$, so that

$$\begin{aligned} G_{Mat'}(\eta, w) = & \frac{(-a)^r}{(1+a)^{r'-1}} \left(\prod_{j=1}^{s'-1} \left(\frac{a_j - a}{(1+a)(1+a_j)} \right)^{-r_j} \right) \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2} \right) M} \right)^{1-t'} \\ & e^{-(1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2} \right) M} \eta w - \frac{1}{2} (1+a)^2 \left(1 - \frac{1}{\gamma^2 a^2} \right) M w^2} \frac{dw}{w^{t'}} (1 + \mathcal{O}(M^{-1/2})), \end{aligned} \quad (5.142)$$

and the $\mathcal{O}(M^{-1/2})$ is independent of η . After the substitution $u = (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2} \right) M} w$, we get

$$\frac{1}{2\pi i} \oint_{\tilde{\Gamma}_0^M} G_{Mat'}(\eta, w) dw = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_\infty} e^{-\eta u - \frac{u^2}{2}} \frac{du}{u^{t'}} (1 + \mathcal{O}(M^{-1/2})). \quad (5.143)$$

On $\tilde{\Gamma}_\infty$, if $\eta \geq T$, $|e^{-(\eta-T)u}| \leq e^{|T/3|} e^{\eta/3}$, and we have

$$\begin{aligned} \left| \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_0^M} G_{Mat'}(\eta, w) dw - \frac{(-a)^r}{(1+a)^{r'-1} \bar{C}_{r'-t'}} \frac{(-1)^{t'-1}}{(t'-1)!} H_{t'-1}(\eta) \right| \leq \\ e^{|T/3|} e^{\eta/3} \left| \frac{(-a)^r}{(1+a)^{r'-1} \bar{C}_{r'-t'}} \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_\infty} \left| e^{-\eta u - \frac{u^2}{2}} u^{-t'} \right| du \mathcal{O}(M^{-1/2}) \right|, \end{aligned} \quad (5.144)$$

and we can get the result by direct calculation. \square

Now we estimate the integral of $G_{Mat'}$ over $\tilde{\Gamma}_{\text{right}}^M$, where $\tilde{\Gamma}_{\text{right}}^M$ is defined as the

union of $\tilde{\Gamma}_1^M \cup \dots \cup \tilde{\Gamma}_8^M$, as

$$\tilde{\Gamma}_1^M = \left\{ \delta - it \left| -\frac{\delta}{3} \leq t \leq \frac{\delta}{3} \right. \right\}, \quad (5.145)$$

$$\tilde{\Gamma}_2^M = \left\{ -t + \frac{\delta}{3}i \left| -\frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} \leq t \leq -\delta \right. \right\}, \quad (5.146)$$

$$\tilde{\Gamma}_3^M = \left\{ t - \frac{\delta}{3}i \left| \delta \leq t \leq \frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} \right. \right\}, \quad (5.147)$$

$$\tilde{\Gamma}_4^M = \left\{ \frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} - it \left| -4 \leq t \leq -\frac{\delta}{3} \right. \right\}, \quad (5.148)$$

$$\tilde{\Gamma}_5^M = \left\{ \frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} - it \left| \frac{\delta}{3} \leq t \leq 4 \right. \right\}, \quad (5.149)$$

$$\tilde{\Gamma}_6^M = \left\{ -t + 4i \left| -C_{\text{right}} \leq t \leq -\frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} \right. \right\}, \quad (5.150)$$

$$\tilde{\Gamma}_7^M = \left\{ t - 4i \left| \frac{\gamma^2 a^2 - 1}{(1 + \gamma^2 a)(1 + a)} \leq t \leq C_{\text{right}} \right. \right\}, \quad (5.151)$$

$$\tilde{\Gamma}_8^M = \{ C_{\text{right}} + it \mid -4 \leq t \leq 4 \}, \quad (5.152)$$

where δ is a very small positive number and C_{right} is the same large positive number as that in the definition of $\bar{\Gamma}^M$. Then we can prove

Lemma 5.13. *If T is fixed and M is large enough,*

$$\left| \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_{\text{right}}^M} G_{Mat'}(\eta, w) dw \right| < \frac{1}{2} \frac{e^{\eta/3}}{M^{1/3}}, \quad (5.153)$$

for any $\eta \geq T$.

The proof is similar to those of lemmas 5.1 and 5.2.

Hence by lemmas 5.12 and 5.13, and (5.140), we have

$$\left| \left((1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} \right)^{1-t'} (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a}y} e^{-2\eta/3} \tilde{\psi}_{r'}(p+q\eta) - \frac{(-a)^r}{(1+a)^{r'-1} \bar{C}_{a,r'-t'}} \frac{(-1)^{t'-1}}{(t'-1)!} e^{-2\eta/3} H_{t'-1}(\eta) \right| < \frac{e^{-\eta/3}}{M^{1/3}}. \quad (5.154)$$

For $\tilde{\psi}_{r'}$ with $a_{s'} < a$ or $\tilde{\psi}$, we can choose $\tilde{\Gamma}_{\text{right}}^M$ so that all poles are included in it, and we can get the estimation by results similar to lemma 5.13. The result is

$$\left| (1+a) \sqrt{\left(1 - \frac{1}{\gamma^2 a^2}\right) M} (-a)^N (1+a)^{M-N} e^{\frac{M}{1+a}y} e^{-2\eta/3} \tilde{\psi}_*(p+q\eta) \right| < \frac{e^{-\eta/3}}{M^{1/3}}, \quad (5.155)$$

where $\tilde{\psi}_*$ stands for $\tilde{\psi}_{r'}$ with $a_{s'} < a$, or simply $\tilde{\psi}$.

5.4 Asymptotics of Laguerre polynomials and related functions

In chapter 3, we need asymptotic results of $L_{2N-2}^{(2(M-N))}$, $L_{2N-1}^{(2(M-N))}$, ψ_{2N-1} , ψ'_{2N-1} , etc.

They can be expressed by linear combination of Laguerre polynomials and has the similar integral representation. Like (2.41), we have

$$L_{2N-1}^{(2(M-N))}(Mx) = \frac{e^{2Mx}}{2\pi i} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M-1}}{(z-1)^{2N}} dz, \quad (5.156)$$

and by (3.54), we get

$$\begin{aligned}
\varphi_{2N-1}(x) &= e^{\frac{a}{1+a}2Mx} - \frac{(1+a)^{2(M-N)+1}}{2\pi i} e^{2Mx} \oint_{\Gamma} e^{-2Mxz} \frac{1 + \left(a\frac{z}{z-1}\right)^{2N-1}}{1 + a\frac{z}{z-1}} \frac{z^{2(M-N)}}{z-1} dz \\
&= e^{\frac{a}{1+a}2Mx} - \frac{(1+a)^{2(M-N)+1}}{2\pi i} e^{2Mx} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2(M-N)}}{z - \frac{1}{1+a}} dz \\
&\quad - \frac{(1+a)^{2(M-N)+1} a^{2N-1}}{2\pi i} e^{2Mx} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M}}{(z-1)^{2N}} \frac{z-1}{((1+a)z-1)z} dz.
\end{aligned} \tag{5.157}$$

If the pole $z = \frac{1}{1+a}$ is inside of Γ , then

$$\frac{(1+a)^{2(M-N)+1}}{2\pi i} e^{2Mx} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2(M-N)}}{z - \frac{1}{1+a}} dz = e^{\frac{a}{1+a}2Mx}, \tag{5.158}$$

and

$$\varphi_{2N-1}(x) = -\frac{(1+a)^{2(M-N)+1} a^{2N-1}}{2\pi i} e^{2Mx} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M}}{(z-1)^{2N}} \frac{z-1}{((1+a)z-1)z} dz. \tag{5.159}$$

All other relevant functions can also be expressed by integrals by (2.41). We analyze three typical examples, and all other results can be derived similarly.

5.4.1 $L_{2N-1}^{(2(M-N))}(2Mx)$ around $(1 + \gamma^{-1})^2$

We assume in $p = (1 + \gamma^{-1})^2$ and $q = \frac{(\gamma+1)^{4/3}}{\gamma(2M)^{2/3}}$, and take $x = p + q\xi$. By methods in the analysis of $\tilde{\psi}(p + q\xi)$ and $\tilde{\psi}_{r'}(p + q\xi)$ in subsection 5.1.2, if we denote

$$\tilde{\psi}(\xi) = \frac{1}{2\pi i} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M-1}}{(z-1)^{2N}} dz, \tag{5.160}$$

we have for $\xi \geq T$

$$\left| \frac{(\gamma+1)^{4/3}}{\gamma} (2M)^{1/3} \frac{(\gamma+1)^{2(M-N)-1}}{\gamma^{2M}} e^{\frac{\gamma}{\gamma+1} 2Mx} \tilde{\psi}(\xi) - (-1) \text{Ai}(\xi) \right| < \frac{e^{-\xi/2}}{M^{1/40}}. \quad (5.161)$$

The relation between $\tilde{\psi}(\xi)$ and $L_{2N-1}^{(2(M-N))}(2Mx)$ is

$$L_{2N-1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} = \frac{x^{M-N-1/2}}{e^{\frac{\gamma-1}{\gamma+1} Mx}} e^{\frac{\gamma}{\gamma+1} 2Mx} \tilde{\psi}(\xi). \quad (5.162)$$

For $x = p + q\xi$, $\xi \geq T$, we have the point-wise with respect to ξ

$$\lim_{M \rightarrow \infty} \frac{e^{M-N}}{\left(\frac{\gamma+1}{\gamma}\right)^{2(M-N)-1}} \frac{x^{M-N-1/2}}{e^{\frac{\gamma-1}{\gamma+1} Mx}} = 1, \quad (5.163)$$

and for $\xi \geq 0$,

$$0 < \frac{e^{M-N}}{\left(\frac{\gamma+1}{\gamma}\right)^{2(M-N)-1}} \frac{x^{M-N-1/2}}{e^{\frac{\gamma-1}{\gamma+1} Mx}} \leq 1. \quad (5.164)$$

Therefore we know that for M large enough and $\xi \geq T$

$$\left| \gamma^{-2N-1} (\gamma+1)^{4/3} (2M)^{1/3} e^{M-N} L_{2N-1}^{(2(M-N))}(2Mx) x^{M-N-1/2} e^{-Mx} - (-1) \text{Ai}(\xi) \right| < \frac{e^{-\xi/2}}{M^{1/40}}. \quad (5.165)$$

5.4.2 $\psi_{2N-1}(x)$ around $(1 + \gamma^{-1})^2$ when $a = \gamma^{-1}$

We still take $p = (1 + \gamma^{-1})^2$, $q = \frac{(\gamma+1)^{4/3}}{\gamma(2M)^{2/3}}$, and $z = p + q\xi$. We use the integral representation (5.159) of $\psi_{2N-1}(x)$, and we will make sure that the pole $z = \frac{1}{1+a}$ is inside Γ when we deform it. Since $a = \gamma^{-1}$, we can write (5.159) as

$$\psi_{2N-1}(x) = -\frac{(\gamma+1)^{2(M-N)+1}}{\gamma^{2M}} \frac{e^{2Mx}}{2\pi i} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M}}{(z-1)^{2N}} \frac{z-1}{((1+\gamma^{-1})z-1)z} dz. \quad (5.166)$$

If we denote

$$\tilde{\psi}_{2N-1}(\xi) = \frac{1}{2\pi i} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M}}{(z-1)^{2N}} \frac{z-1}{((1+\gamma^{-1})z-1)z} dz, \quad (5.167)$$

by methods in the analysis of $\tilde{\psi}_{r'}(p+q\xi)$ in subsection 5.2.2, we have for $\xi \geq T$,

$$\left| \frac{(\gamma+1)^{2(M-N)+1}}{\gamma^{2M}} e^{\frac{\gamma}{\gamma+1}2Mx} \tilde{\psi}_{2N-1}(\xi) - (-1)s^{(1)}(\xi) \right| < \frac{e^{\xi/6}}{M^{1/40}}. \quad (5.168)$$

We can express $\psi_{2N-1}(x)$ as

$$\psi_{2N-1}(x) = \frac{x^{M-N+1/2}}{e^{\frac{\gamma-1}{\gamma+1}Mx}} \frac{(\gamma+1)^{2(M-N)+1}}{\gamma^{2M}} e^{\frac{\gamma}{\gamma+1}2Mx} \tilde{\psi}_{2N-1}(\xi). \quad (5.169)$$

Similar to (5.163) and (5.164), we have point wise convergence with respect to ξ

$$\lim_{M \rightarrow \infty} \frac{e^{M-N}}{\left(\frac{\gamma+1}{\gamma}\right)^{2(M-N)+1}} \frac{x^{M-N+1/2}}{e^{\frac{\gamma-1}{\gamma+1}Mx}} = 1, \quad (5.170)$$

and if $\xi \geq T$, then

$$0 < \frac{e^{M-N}}{\left(\frac{\gamma+1}{\gamma}\right)^{2(M-N)+1}} \frac{x^{M-N+1/2}}{e^{\frac{\gamma-1}{\gamma+1}Mx}} < 1 + \mathcal{O}\left(\frac{1}{M}\right). \quad (5.171)$$

Therefore, we have

$$\left| e^{M-N} \left(\frac{\gamma}{\gamma+1}\right)^{2(M-N)+1} \psi_{2N-1}(x) - s^{(1)}(\xi) \right| < \frac{e^{\xi/6}}{M^{1/40}}. \quad (5.172)$$

5.4.3 $L_{2N-1}^{(2(M-N))}(2Mx)$ around $(1+a)\left(1+\frac{1}{\gamma^2 a}\right)$ when $a > \gamma^{-1}$

We take $p = (1+a)\left(1+\frac{1}{\gamma^2 a}\right)$, $q = (1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}}\frac{1}{\sqrt{2M}}$ and $x = p + q\xi$. Like (5.160), we define

$$\tilde{\psi}_a(\xi) = \frac{1}{2\pi i} \oint_{\Gamma} e^{-2Mxz} \frac{z^{2M-1}}{(z-1)^{2N}} dz. \quad (5.173)$$

With $g(z)$ defined in (5.113), we have

$$\tilde{\psi}_a(\xi) = \frac{1}{2\pi i} \oint_{\Gamma} e^{2Mg(z)-(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)}2M\xi z} \frac{dz}{z}. \quad (5.174)$$

After the substitution $z = w + \frac{\gamma^2 a}{1+\gamma^2 a}$, we get similar to (5.117),

$$\begin{aligned} & \oint_{\Gamma} e^{2Mg(z)-(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)}2M\xi z} \frac{dz}{z} \\ &= \oint_{\tilde{\Gamma}^M} e^{-2M\left((1+a)+(1-\gamma^{-2})\log(1+\gamma^2 a)-\log(\gamma 62a-\gamma^{-2}\pi i \frac{1}{2}(\gamma^{-1}+\gamma a)^2\left(1-\frac{1}{\gamma^2 a^2}\right)w^2-R_3(w)\right)} \\ & \quad e^{-(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)}2M\xi\left(w+\frac{\gamma^2 a}{1+\gamma^2 a}\right)} \frac{dw}{w+\frac{\gamma^2 a}{1+\gamma^2 a}} \\ &= \frac{(\gamma^2 a)^{2M} e^{-\frac{\gamma^2 a}{1+\gamma^2 a}2Mx}}{(1+\gamma^2 a)^{2(M-N)}} \\ & \quad \oint_{\tilde{\Gamma}^M} e^{(\gamma^{-1}+\gamma a)^2\left(1-\frac{1}{\gamma^2 a^2}\right)Mw^2-(1+a)\sqrt{\left(1-\frac{1}{\gamma^2 a^2}\right)}2M\xi w+2MR_3(w)} \frac{dw}{w+\frac{\gamma^2 a}{1+\gamma^2 a}}, \end{aligned} \quad (5.175)$$

where

$$R_3(w) = \mathcal{O}(w^3), \text{ as } w \rightarrow 0, \quad (5.176)$$

and $\tilde{\Gamma}^M$ is a contour around $w = \frac{1}{1+\gamma^2 a}$, composed of $\tilde{\Gamma}_1^M$, $\tilde{\Gamma}_2^M$, $\tilde{\Gamma}_3^M$ and $\tilde{\Gamma}_4^M$, which are

defined as

$$\tilde{\Gamma}_1^M = \{-it + q^{-1}/(2M) \mid -2 \leq t \leq 2\}, \quad (5.177)$$

$$\tilde{\Gamma}_2^M = \{4 - t + 2i \mid 0 \leq t \leq 4 - q^{-1}/M\}, \quad (5.178)$$

$$\tilde{\Gamma}_3^M = \{4 + it \mid -2 \leq t \leq 2\}, \quad (5.179)$$

$$\tilde{\Gamma}_4^M = \{t - 2i \mid q^{-1}/(2M) \leq t \leq 4\}. \quad (5.180)$$

By methods in the analysis of $\psi_{r'}(p + q\xi)$ in subsection 5.3.1, we have for $\xi \geq T$,

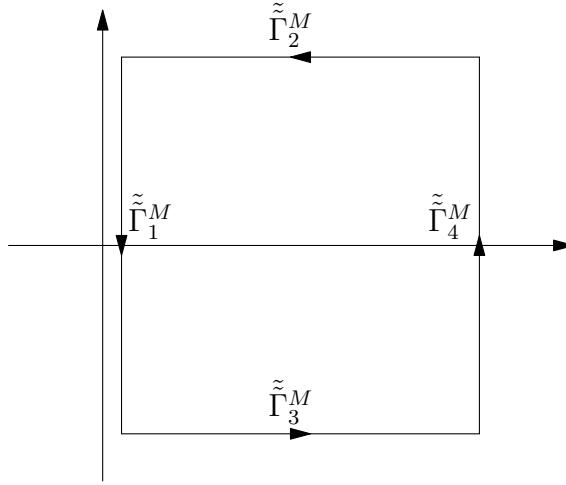


Figure 5.11: $\tilde{\Gamma}^M$

$$\left| \frac{(1 + \gamma^2 a)^{2(M-N)}}{(\gamma^2 a)^{2M}} \sqrt{(\gamma^2 a^2 - 1)2M} e^{\frac{\gamma^2 a}{1+\gamma^2 a} 2Mx} \tilde{\psi}_a(\xi) - \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\gamma + \gamma a}{1 + \gamma^2 a} \xi \right)^2} \right| < \frac{e^{-\xi}}{M^{1/10}}. \quad (5.181)$$

We have the result similar to (5.162)

$$L_{2N-1}^{(2(M-N))}(Mx) x^{M-N-1/2} e^{-\frac{\gamma^2 a + a + 2}{(a+1)(\gamma^2 a + 1)} Mx} = \frac{x^{M-N-1/2}}{e^{\frac{M-N}{(1+a)(1+\frac{1}{\gamma^2 a})} x}} e^{\frac{\gamma^2 a}{1+\gamma^2 a} 2Mx} \tilde{\psi}_a(\xi). \quad (5.182)$$

For $x = p + q\xi$, $\xi \geq T$, we have the pointwise convergence with respect to ξ

$$\lim_{M \rightarrow \infty} \frac{e^{M-N}}{\left((1+a) \left(1 + \frac{1}{\gamma^2 a}\right)\right)^{M-N-1/2}} \frac{x^{M-N-1/2}}{e^{\frac{M-N}{(1+a) \left(1 + \frac{1}{\gamma^2 a}\right)} x}} = e^{-\frac{1}{4} \frac{(\gamma^2 a^2 - 1)(\gamma^2 - 1)}{(\gamma^2 a + 1)^2} \xi^2}, \quad (5.183)$$

and for $\xi \geq 0$,

$$0 < \frac{e^{M-N}}{\left((1+a) \left(1 + \frac{1}{\gamma^2 a}\right)\right)^{M-N-1/2}} \frac{x^{M-N-1/2}}{e^{\frac{M-N}{(1+a) \left(1 + \frac{1}{\gamma^2 a}\right)} x}} < 1. \quad (5.184)$$

Therefore if M is large enough,

$$\left| \frac{(\gamma^2 a + 1)^{M-N+1/2} e^{M-N} \sqrt{(\gamma^2 a^2 - 1)2M}}{(\gamma^2 a)^{M+N+1/2} (a+1)^{M-N-1/2}} L_{2N-1}^{(2(M-N))}(Mx) x^{M-N-1/2} e^{-\frac{\gamma^2 a + a + 2}{(a+1)(\gamma^2 a + 1)} Mx} \right. \\ \left. - \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{4} \frac{\gamma^4 a^2 + \gamma^2 a^2 + 4\gamma^2 a + \gamma^2 + 1}{(\gamma^2 a + 1)^2} \xi^2} \right| < \frac{e^{-\xi}}{M^{1/10}}. \quad (5.185)$$

Bibliography

- [1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.
- [2] M. Adler, P. J. Forrester, T. Nagao, and P. van Moerbeke. Classical skew orthogonal polynomials and random matrices. *J. Statist. Phys.*, 99(1-2):141–170, 2000.
- [3] T. W. Anderson. Asymptotic theory for principal component analysis. *Ann. Math. Statist.*, 34:122–148, 1963.
- [4] Steen Andersson. Invariant normal models. *Ann. Statist.*, 3:132–154, 1975.
- [5] Jinho Baik. Painlevé formulas of the limiting distributions for nonnull complex sample covariance matrices. *Duke Math. J.*, 133(2):205–235, 2006.
- [6] Jinho Baik, Gérard Ben Arous, and Sandrine Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5):1643–1697, 2005.
- [7] Pavel M. Bleher and Arno B. J. Kuijlaars. Integral representations for multiple Hermite and multiple Laguerre polynomials. *Ann. Inst. Fourier (Grenoble)*, 55(6):2001–2014, 2005.
- [8] N. G. de Bruijn. On some multiple integrals involving determinants. *J. Indian Math. Soc. (N.S.)*, 19:133–151 (1956), 1955.
- [9] P. J. Forrester. The spectrum edge of random matrix ensembles. *Nuclear Phys. B*, 402(3):709–728, 1993.
- [10] P. J. Forrester. Painlevé transcendent evaluation of the scaled distribution of the smallest eigenvalue in the laguerre orthogonal and symplectic ensembles. <http://www.citebase.org/abstract?id=oai:arXiv.org:nlin/0005064>, 2000.

- [11] P. J. Forrester, T. Nagao, and G. Honner. Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges. *Nuclear Phys. B*, 553(3):601–643, 1999.
- [12] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.
- [13] N. R. Goodman. Statistical analysis based on a certain multivariate complex Gaussian distribution. (An introduction). *Ann. Math. Statist.*, 34:152–177, 1963.
- [14] Philip J. Hanlon, Richard P. Stanley, and John R. Stembridge. Some combinatorial aspects of the spectra of normally distributed random matrices. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 151–174. Amer. Math. Soc., Providence, RI, 1992.
- [15] Kurt Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.
- [16] Iain M. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist.*, 29(2):295–327, 2001.
- [17] Peter D. Lax. *Functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2002.
- [18] I. G. Macdonald. Commuting differential operators and zonal spherical functions. In *Algebraic groups Utrecht 1986*, volume 1271 of *Lecture Notes in Math.*, pages 189–200. Springer, Berlin, 1987.
- [19] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [20] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues in certain sets of random matrices. *Mat. Sb. (N.S.)*, 72 (114):507–536, 1967.
- [21] Madan Lal Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.
- [22] Robb J. Muirhead. *Aspects of multivariate statistical theory*. John Wiley & Sons Inc., New York, 1982. Wiley Series in Probability and Mathematical Statistics.
- [23] Debashis Paul. Asymptotics of the leading sample eigenvalues for a spiked covariance model. *Statist. Sinica*, to appear.

- [24] Michael Prähofer and Herbert Spohn. Current fluctuations for the totally asymmetric simple exclusion process. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 185–204. Birkhäuser Boston, Boston, MA, 2002.
- [25] Jirō Sekiguchi. Zonal spherical functions on some symmetric spaces. In *Proceedings of the Oji Seminar on Algebraic Analysis and the RIMS Symposium on Algebraic Analysis (Kyoto Univ., Kyoto, 1976)*, volume 12, pages 455–459, 1976/77 supplement.
- [26] Richard P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.*, 77(1):76–115, 1989.
- [27] Gábor Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.
- [28] Craig A. Tracy and Harold Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159(1):151–174, 1994.
- [29] Craig A. Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177(3):727–754, 1996.
- [30] Craig A. Tracy and Harold Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *J. Statist. Phys.*, 92(5-6):809–835, 1998.
- [31] Craig A. Tracy and Harold Widom. Matrix kernels for the Gaussian orthogonal and symplectic ensembles. *Ann. Inst. Fourier (Grenoble)*, 55(6):2197–2207, 2005.